Math378 **Quiz 12** **Wed. Apr 20, 2022**

Name: Solutions

Directions: Show all work.

1. **[3 points]** Suppose that $A(x) = \frac{1}{(1-2x)^8}$. What is the coefficient of x^{12} in $A(x)$?

$$A(x) = \sum_{n\ge 0} \underbrace{\qquad\quad}_{\#\text{solns to } x_1+\cdots+x_8=n} (2x)^n = \sum_{n\ge 0}\binom{n+7}{7}(2x)^n = \sum_{n\ge 0}\binom{n+7}{7}\cdot 2^n\cdot x^n.$$

So coeff. of x^{12} is $\binom{12+7}{7}\cdot 2^{12} = \boxed{\binom{19}{7}2^{12}}$.

2. **[4 points]** Suppose that $(a_n)_{n\ge 0}$ has OGF $A(x) = \frac{1}{(1-2x)(1-3x)}$. Find a formula for a_n.

$$\frac{1}{(1-2x)(1-3x)} = \frac{A}{1-2x} + \frac{B}{1-3x} \implies A(1-3x) + B(1-2x) = 1$$

$x=\frac{1}{3}$: $B(1-\frac{2}{3}) = 1$, $B(\frac{1}{3}) = 1$, $B=3$

$x=\frac{1}{2}$: $A(1-\frac{3}{2}) = 1$, $A(-\frac{1}{2}) = 1$, $A=-2$

So $A(x) = (-2)\cdot\frac{1}{1-2x} + (3)\cdot\frac{1}{1-3x} = -2\sum_{n\ge 0}(2x)^n + 3\sum_{n\ge 0}(3x)^n$

So a_n = coeff of x^n in $A(x) = -2\cdot 2^n + 3\cdot 3^n = \boxed{3^{n+1} - 2^{n+1}}$

3. **[3 points]** Let $a_0 = 2$, $a_1 = 1$, and $a_n = 3a_{n-1} + a_{n-2} + 2^n$ for $n \ge 2$. Find the OGF $A(x)$.

$$\sum_{n\ge 2} a_n x^n = 3\sum_{n\ge 2} a_{n-1}x^n + \sum_{n\ge 2} a_{n-2}x^n + \sum_{n\ge 2} 2^n x^n$$

$$A(x) - 2 - 1x = 3x\sum_{n\ge 2} a_{n-1}x^{n-1} + x^2\sum_{n\ge 2} a_{n-2}x^{n-2} + \left(\sum_{n\ge 0}(2x)^n - 1 - 2x\right)$$

$$A(x) - 2 - x = 3x(A(x) - 2) + x^2A(x) + \frac{1}{1-2x} - 1 - 2x$$

$$A(x) - 2 - x = 3xA(x) - 6x + x^2A(x) + \frac{1}{1-2x} - 1 - 2x$$

$$[1 - 3x - x^2]A(x) = 2 + x - 6x + \frac{1}{1-2x} - 1 - 2x$$

$$[1 - 3x - x^2]A(x) = 1 - 7x + \frac{1}{1-2x}$$

$$A(x) = \frac{1-7x}{1-3x-x^2} + \frac{1}{(1-2x)(1-3x-x^2)} = \frac{(1-7x)(1-2x)+1}{(1-2x)(1-3x-x^2)} = \boxed{\frac{2-9x+14x^2}{(1-2x)(1-3x-x^2)}}$$