Name: Solutions

Directions: Show all work.

1. **[2 parts, 3 points each]** Binomial Theorem.

   (a) Use the binomial theorem to expand $(x^2+1)^n$.

$$(x^2+1)^n = \sum_{k=0}^{n} \binom{n}{k} (x^2)^k (1)^{n-k}$$

$$= \boxed{\sum_{k=0}^{n} \binom{n}{k} x^{2k}}$$

   (b) Differentiate both sides of part (a) to find a formula for $\sum_{k=1}^{n} 2k\binom{n}{k}3^{2k-1}$.

$$\frac{d}{dx}\left[(x^2+1)^n\right] = \frac{d}{dx}\left[\sum_{k=0}^{n}\binom{n}{k}x^{2k}\right]$$

$$n(x^2+1)^{n-1}\cdot 2x = \sum_{k=0}^{n} 2k\binom{n}{k}x^{2k-1} = \sum_{k=1}^{n} 2k\binom{n}{k}x^{2k-1}$$

With $x=3$: $$\sum_{k=1}^{n} 2k\binom{n}{k}3^{2k-1} = n(3^2+1)^{n-1}\cdot 2(3) = \boxed{6n(10)^{n-1}}$$

2. **[4 points]** How many integer solutions are there to $x_1+\ldots+x_8=50$ such that $0 \le x_i \le 5$ for each i? Use inclusion/exclusion to give a summation formula.

Let $\mathcal{U}$ = set of non-neg solns to $x_1+\cdots+x_8=50$

For $1 \le i \le 8$, let A_i be the solutions in $\mathcal{U}$ with $x_i \ge 6$.

Note: if $S \subseteq [8]$ and $|S|=k$, then $\left|\bigcap_{j\in S} A_j\right|$ = # solns to $\hat{x}_1+\cdots+\hat{x}_8 = 50-6k$

$\Rightarrow$ $50-6k$ stars, 7 bars

$\Rightarrow \binom{(50-6k)+(7)}{7} = \binom{57-6k}{7}$

So # solns $= \left|\overline{\bigcup_{i=1}^{8} A_i}\right| = \sum_{S\subseteq[8]} (-1)^{|S|}\left|\bigcap_{j\in S}A_j\right| = \boxed{\sum_{k=0}^{8}\binom{8}{k}(-1)^k\binom{57-6k}{7}}$