Name: Solutions

Directions: Show all work.

1. [**5 points**] Prove for $n \geq 0$, we have $\sum_{k=0}^{n} k2^k = (n-1)2^{n+1} + 2$.

By induction on n. If $n=0$, then the LHS is $0 \cdot 2^0$, or 0. The RHS is $(-1)2+2$, which also equals zero.

Suppose that $n \geq 1$. By the induction hypothesis, we have

$$\sum_{k=0}^{n-1} k2^k = ((n-1)-1)2^{(n-1)+1} + 2 = (n-2)2^n + 2.$$

Adding $n2^n$ to both sides gives

$$\sum_{k=0}^{n} k2^k = (n-2)2^n + 2 + n \cdot 2^n = (2n-2) \cdot 2^n + 2 = (n-1)2^{n+1} + 2$$

It follows that the identity holds at n. $\blacksquare$

2. [**5 points**] Prove that for each integer n with $n \geq 2$, we have that $2^n + 3^n < 4^n$.

By induction on n. If $n=2$, then we have $2^2+3^2 = 4+9 = 13 < 16 = 4^2$, and so the inequality holds.

Suppose $n \geq 3$. By the induction hypothesis, we have that $4^{n-1} > 2^{n-1} + 3^{n-1}$.

Multiplying both sides by 4 gives

$$\begin{aligned} 4^n &> 4 \cdot 2^{n-1} + 4 \cdot 3^{n-1} \\ &> 2 \cdot 2^{n-1} + 3 \cdot 3^{n-1} \\ &= 2^n + 3^n \end{aligned}$$

It follows that the inequality also holds at n. $\blacksquare$