Directions: Solve the following problems. All written work must be your own. See the course syllabus for detailed rules.

1. [9.1.22] Let $A(x) = \frac{1}{1+x+x^2}$, where $A(x)$ is the OGF for the sequence $(a_n)_{n\geq 0}$.

   (a) Find a recurrence relation for a_n. Hint: try to undo the steps one would take to find the generating function from a recurrence.

   (b) Find $a_0, \ldots, a_5$.

   (c) Let α and β be the roots of $x^2 + x + 1$. Even though α and β are complex numbers, find constants C and D such that $A(x) = \frac{1}{(x-\alpha)(x-\beta)} = \frac{C}{x-\alpha} + \frac{D}{x-\beta}$.

   (d) Find a closed formula for a_n.

2. [9.2.8] Let n be a non-negative integer. You have an unlimited supply of oranges, apples, bananas, and pears, and you are going to make bags consisting of n of these fruits. However, you insist that the numbers of apples and oranges must be odd. Let a_n be the number of possible such bags.

   (a) What is the ordinary generating function for the sequence $(a_n)_{n\geq 0}$?

   (b) Find a partial fraction decomposition for this generating function.

   (c) Find a formula for a_n.

   (d) What is a_{13}?

3. [10.7.4] Let G be a graph. Prove that $\chi(G) \leq 2$ if and only if G is bipartite.

4. [10.7.5] Let G be an n-vertex graph. Prove that $\alpha(G) \cdot \chi(G) \geq n$.

5. Chromatic number and odd cycles.

   (a) Let G be a graph such that $\chi(G) \geq 9$. Prove that G has three vertex-disjoint odd cycles.

   (b) Find an example of a graph G with $\chi(G) = 8$ such that G does not contain three vertex-disjoint odd cycles.