Directions: Solve 5 of the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. Applications of Inclusion/Exclusion.

   (a) A mathematics department has n professors and $2n$ courses; each professor teaches two courses each semester. How many ways are there to assign the courses in the fall semester? How many ways are there to assign them in the spring so that no professor teaches the same two courses in the spring as in the fall? (Your answer for the spring semester may be a summation.)

   (b) At a circular table are n students taking an exam. The exam has four versions. Given that no two neighboring students have the same version, how many ways are there to assign the exams? Do not leave the answer as a sum.

2. Let M be an $(n \times n)$-Latin square that has the block form $\left(\begin{smallmatrix} X & Y \\ Y & X \end{smallmatrix}\right)$, where X and Y are are Latin squares of odd order. Prove that M has no *transversal*, where a transversal is a set of n cells in distinct rows and columns having distinct values. Use this to prove that there is no Latin square orthogonal to M.

3. Let $f(n)$ be the least k such that every set of k elements in $[n]$ has two disjoint subsets with the same sum. Prove that $1 + \lfloor \lg n \rfloor < f(n) \le \lceil 1 + \lg n + \lg \lg n \rceil$ for sufficiently large n. (Here, $\lg n = \log_2 n$.) (Hint: for the upper bound, show that if $2^k > nk + 1$, then $f(n) \le k$.)

4. A family $\mathcal{F}$ of permutations of $[n]$ is *intersecting* if π and π' take the same value on at least one $k \in [n]$ when $\pi, \pi' \in \mathcal{F}$. Determine the maximum size of an intersecting family of permutations of $[n]$. (Note: correct solutions have two parts. First, an upper bound on the size of intersecting families is needed. Next, an existence proof (usually a construction) is needed to show that some intersecting family meets the upper bound.)

5. *Uniqueness of maximum k-uniform intersecting families.* Let $n \ge 3k$ and let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a maximum intersecting family.

   (a) Let σ be a cyclic ordering of $[n]$, and let $\mathcal{F}_\sigma$ be the subfamily of sets $A \in \mathcal{F}$ such that A is an interval of σ. Prove that σ has an interval I_σ of size $2k-1$ such that $\mathcal{F}_\sigma$ is the set of all the subintervals of I_σ of size k. Conclude that the center x_σ of I_σ belongs to each set in $\mathcal{F}_\sigma$.

   (b) Let σ be a cyclic ordering of $[n]$, and let I_σ and x_σ be as in part (a). Prove that if λ is obtained from σ by swapping consecutive elements $\{y, y'\}$ with $x_\sigma \notin \{y, y'\}$, then $x_\sigma = x_\lambda$. Conclude that there is a single $x \in [n]$ such that $x = x_\sigma$ for each σ.

   (c) With x as in part (b), show that $\mathcal{F}$ is the star $\{A \in \binom{[n]}{k} \colon\, x \in A\}$.

6. A *conical rational combination* of real numbers $r_1, \ldots, r_t$ is a number of the form $\sum_{k=1}^{t} \alpha_k r_k$, where each α_k is a non-negative rational. When $\alpha_k = 0$ for each k, the combination is *trivial*; otherwise, it is *non-trivial.* A set T of real numbers is *good* if every non-trivial positive rational combination of T is irrational.

   (a) Prove that if S is a set of $2n+1$ irrational numbers, then S contains a good subset of size $n+1$. [Hint: show that if T is a maximal good subset, then the complement $S - T$ is also good.]

   (b) Give an example of a set of $2n+1$ irrational numbers with no good subset of size $n+2$.