Directions: Solve 5 of the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. Let a_n be the number of n-tuples in $[4]^n$ that have at least one 1 and have no 2 appearing before the first 1 (note that $\langle a \rangle$ begins $0, 1, 6, \ldots$). Obtain and solve a recurrence for $\langle a \rangle$. Give a direct counting argument (without using summations) to prove the resulting simple formula.

2. A row of n lightbulbs must be turned on; initially, they are all off. Bulb 1 can be turned on or off at any time. For $i > 1$, bulb i can be turned on or off only when bulb $i-1$ is on and all earlier bulbs are off. Let a_n be the number of steps needed to turn on all the lights; note that $\langle a \rangle$ begins $(0, 1, 2, 5, 10, 21, \ldots)$. Let b_n be the number of steps needed to turn on bulb n for the first time.
   (a) Find a recurrence for $\langle b \rangle$ and solve it.
   (b) Use $\langle b \rangle$ to find a recurrence for $\langle a \rangle$.
   (c) Solve the recurrence for $\langle a \rangle$.

3. Let a_n be the number of words of length n of the alphabet $\{0, 1, 2\}$ such that 1 and 2 are never adjacent.
   (a) Obtain a second-order recurrence relation for $\langle a \rangle$. (*Hint*: An easy approach is first find a recurrence having no fixed order. There is also a direct (more difficult) combinatorial argument.)
   (b) Solve for a_n.

4. Generating Functions.
   (a) A child wants to buy candy. Four types of candy have prices two cents, one cent, two cents, and five cents per piece, respectively. Build the enumerator by total cost for the number of ways to spend money. Express the enumerator in the form $1/f(x)$, where $f(x)$ is a finite product of polynomials.
   (b) Let a_n be the number of ways to pick a nonnegative integer r, roll one six-sided die r times, and obtain a list of outcomes with sum n. (Caution: r is not fixed; for example, $(a_0, a_1, a_2, a_3) = (1, 1, 2, 4)$.) Express the generating function for $\langle a \rangle$ as the ratio of two polynomials that each have at most three terms. Obtain a recurrence for $\langle a \rangle$ directly from the generating function. Give a combinatorial argument for the recurrence.

5. For the identity below,
   (a) Give a combinatorial proof by constructing a set that both sides count.
   (b) Use the Binomial Theorem to prove that both sides arise as the coefficient of x^n in the expansion of $(1 + 3x + x^2)^n$. Hint: if you are having trouble generating the LHS with the Binomial Theorem, try to apply your combinatorial argument from (a) in the algebraic context. What plays the role of j in the combinatorial argument? What is the corresponding mechanism in the algebraic context?

$$\sum_{j \geq 0} \binom{n}{j}\binom{2j}{j} = \sum_{k \geq 0} \binom{n}{2k}\binom{2k}{k} 3^{n-2k}$$

6. A gambler and a casino play a game with n blue balls and $n+1$ red balls. The gambler starts with one unit of money (infinitely divisible). At each round, the gambler bets part of his money (possibly 0), and the casino selects a ball (knowing the amount of the gambler's bet). If the ball is blue, then the gambler loses the bet; if it is red, then the gambler gains that amount. The selected ball is discarded. The gambler wants to maximize his money after all the balls are used, while the casino wants to minimize that. If both play optimally, how much does the gambler have at the end? (Hint: Try small cases. Solve a more general problem.)