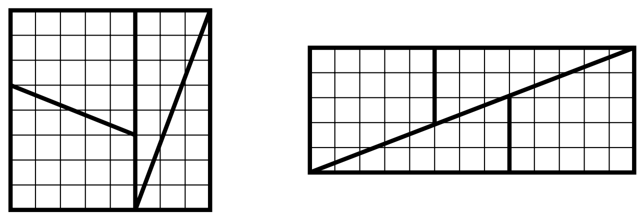
Math571 **Homework 3** **Due March 1, 2019**

Directions: Solve 5 of the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines that apply to all homeworks.

1. A *cyclic shift* of a p-tuple $(x_0, \ldots, x_{p-1})$ is a p-tuple of the form $(x_k, x_{k+1}, \ldots, x_{k+p-1})$, where all indices are taken modulo p. For all non-negative integers a, show that p divides $a^p - a$ using cyclic shifts when p is prime. (Comment: this yields a combinatorial proof of Fermat's Little Theorem.)

2. Prove that $F_n^2 = F_{n-1}F_{n+1} + (-1)^n$ for $n \geq 1$. Manipulate the identity to explain why Lewis Carroll's "proof" below that $64 = 65$ (and larger analogues) seems reasonable.



3. Generating functions.Let $A(x)$ be the generating function for the Fibonacci sequence F_n, with $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.
   (a) Obtain $A(x)$ from the Fibonacci recurrence.
   (b) Obtain $A(x)$ by building it combinatorially (without the recurrence), using the model that F_n is the number of $\{1, 2\}$-lists that sum to n.
   (c) Expand the generating function to prove that $F_n = \sum_{k \geq 0} \binom{n-k}{k}$.

4. Recall that B_n is the number of partitions of $[n]$.
   (a) Prove that for each $k \in [n]$, the inequality $B_n \geq k^{n-k}$ holds.
   (b) Prove that $B_n \leq n!$ for $n \geq 0$.
   (c) Conclude that $\left(\frac{n}{\ln}\right)^{n(1-1/\ln n)} \leq B_n \leq e\sqrt{n+1}\left(\frac{n}{e}\right)^n$.

5. Let $s(n)$ be the number of sequences $(x_1, \ldots, x_k)$ of integers satisfying $1 \leq x_i \leq n$ for all i and $x_i \geq 2x_{i-1}$ for $1 < i \leq k$. (The length of the sequence is not specified, and the empty sequence is included, and therefore $s(0) = 1$.)
   (a) Prove that $s(n) = s(n-1) + s(\lfloor n/2 \rfloor)$ for $n \geq 1$.
   (b) Let $S(t) = \sum_{n \geq 0} s(n)t^n$, so that $S(t)$ is the generating function for the sequence $\langle s \rangle$. Show that $(1-t)S(t) = (1+t)S(t^2)$.

   Medium Challenge: determine good bounds on $s(n)$.

6. Let a_n be the number of domino tilings of a $(2 \times n)$-rectangle and let b_n be the number of domino tilings of a $(3 \times 2n)$-rectangle. Obtain bounded order recurrences for $\langle a \rangle$ and $\langle b \rangle$. (Medium challenge: find a bounded order recurrence for the number of domino tilings of a $(4 \times n)$-rectangle.)