Directions: Solve 5 of the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. Prove the first three identities below by counting a set in two ways. In each case, give a single direct argument without manipulating the formulas. In part (d), find a closed form solution for the sum and give a combinatorial proof.

   (a) $\binom{2n}{n} = 2\binom{2n-1}{n-1}$

   (b) $\sum_k \binom{k}{l}\binom{n}{k} = \binom{n}{l}2^{n-l}$

   (c) $\sum_{i=1}^{n} i(n-i) = \sum_{i=1}^{n}\binom{i}{2}$

   (d) $\sum_{j=1}^{m}(m-j)2^{j-1}$

2. Suppose n is divisible by 6.

   (a) For $0 \le i \le 5$, let S_i be the set of all subsets of $[n]$ with size congruent to i modulo 6. (So S_0 consists of all subsets whose size is divisible by 6, and S_1 collects the sets whose size has remainder 1 when divided by 6, etc.) Prove that $|S_1| = |S_5|$ and $|S_2| = |S_4|$.

   (b) Let $A = |S_0|$, $B = |S_3|$, $C = |S_1| = |S_5|$, and $D = |S_2| = |S_4|$. Show that $A + 2D = 2^{n-1}$ and $B + 2C = 2^{n-1}$.

   (c) Use the Binomial Theorem to show that $3^{n/2}(-1)^{n/6} = (A - B) + (C - D)$.

   (d) Using an appropriate linear combination of the three equations in parts (b) and (c), find a formula $A + C + D$, the number of subsets of $[n]$ whose size is congruent to 0, 1, or 2 modulo 6.

3. By pairing positive and negative contributions, give a combinatorial proof for

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2}\binom{n}{n/2} & \text{if } n \text{ is even} \end{cases}.$$

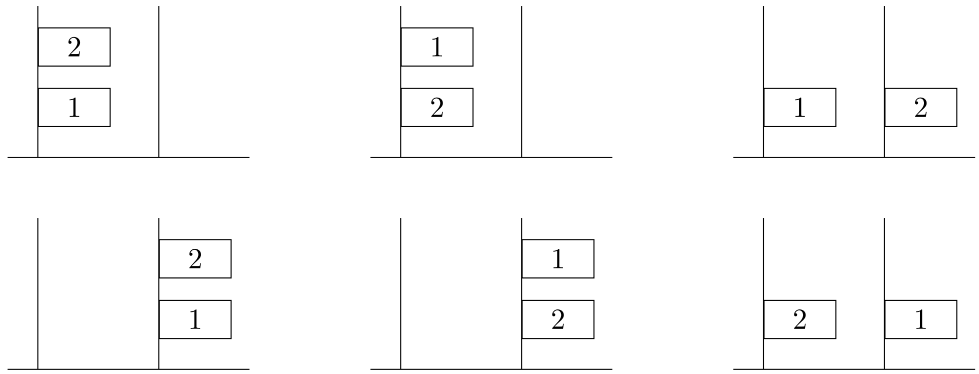
4. Give a combinatorial proof for the following identity by devising a set that both sides count.

$$\sum_{k\ge 1} k\binom{m+1}{r+k+1} = \sum_{i=1}^{m} i2^{i-1}\binom{m-i}{r}$$

5. The line segments from $(j, \ln j)$ to $(j+1, \ln(j+1))$ lie below the curve $y = \ln x$ (since $f(x) = \ln x$ is convex). By comparing the area under the segments from $j = 1$ to $j = n$ with the area under the curve $y = \ln x$ from $x = 1$ to $x = n+1$, show that $n! \le e\sqrt{n+1}(n/e)^n$. [Hint: use that $1 + x \le e^x$ for real x.]

6. Flags on poles.

   (a) Obtain a simple formula for the number of ways to put m distinct flags on a row of r flagpoles. Poles may be empty, and changing the order of flags on a pole changes the arrangement. The formula must only use one "m" and one "r". (The answer is 6 for $m = r = 2$, as shown below.)



(b) Prove that the identity below for rising factorials holds for all $x, y \in \mathbb{R}$.

$$(x+y)^{(n)} = \sum_k \binom{n}{k} x^{(k)} y^{(n-k)}$$