Name: ______________________________

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[12 points]** Suppose that $a \in \mathbb{Z}$. Prove that if a is odd, then $4 \mid a^2 + 2a - 3$.

2. **[12 points]** Suppose that $a \in \mathbb{Z}$. Prove that if $3 \mid a^2 + 5a + 1$, then $3 \nmid a$.

3. **[2 parts, 12 points each]** Irrational numbers.

   (a) Prove that $\sqrt{6}$ is irrational.

   (b) Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

4. **[12 points]** Determine the coefficient of x^3 in $(x+1)^8 - (x-1)^6$ explicitly. No justification/sentences required.

5. **[12 points]** Suppose that $a, b \in \mathbb{Z}$. Use the binomial theorem to prove that $(a+b)^5 \equiv a^5 + b^5 \pmod{5}$.

6. [**12 points**] Critique the following proof. Is it correct? If so, can it be improved? If not, can it be fixed by modifying the proof, modifying the statement of the theorem, or both?

   Theorem 1. *If $x, y \in \mathbb{Z}$, then $x^3 + y^3$ is not prime.*

   Proof: Observe that $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$. Since $x + y$ divides $x^3 + y^3$ and $1 < x + y < x^3 + y^3$, it follows that $x^3 + y^3$ is not prime. $\square$

7. **[16 points]** Suppose that n is an integer. Prove that n is the product of two consecutive integers if and only if $4n + 1$ is an odd square.

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