Name: Solutions.

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[2.5 points]** Let x and y be real numbers. Prove that if x is rational and xy is irrational, then y is irrational.

Suppose for a contradiction that x is rational, xy is irrational, and y is rational. Since x and y are rational, it follows that $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some integers $a, b, c,$ and d with $b \neq 0$ and $d \neq 0$. It follows that $xy = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ and therefore xy is rational. But this contradicts that xy is irrational. $\square$

2. **[2.5 points]** Let a and n be integers. Prove that if $a \mid n$ and $a \mid n+1$, then $a = 1$ or $a = -1$.

Since $a \mid n$ and $a \mid n+1$, we have that $n = ak_1$ and $n+1 = ak_2$ for some $k_1, k_2 \in \mathbb{Z}$. Subtracting the first equation from the second gives $1 = ak_2 - ak_1$, and so $1 = a(k_2 - k_1)$. It follows that $a \mid 1$. Since the only divisors of 1 are 1 and -1, we have that $a = -1$ or $a = 1$. $\square$

3. [2.5 points] Let n be an odd positive integer. Prove that $\sqrt{2n}$ is irrational.

Suppose for a contradiction that $\sqrt{2n}$ is rational, and so $\sqrt{2n} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. By canceling common factors, we may assume that a and b have no common positive divisors besides 1. Squaring both sides gives $2n = \frac{a^2}{b^2}$, and so $2nb^2 = a^2$. Therefore a is even, and so $a = 2k$ for some $k \in \mathbb{Z}$. It follows that $2nb^2 = (2k)^2 = 4k^2$, and so $nb^2 = 2k^2$. Since nb^2 is even and n is odd, it follows that b is even. This is a contradiction, since both a and b are divisible by 2. ☒

4. [2.5 points] Let n be an integer. Prove that $3 \nmid n^2 + 1$.

Suppose for a contradiction that $3 \mid n^2+1$. By the division algorithm we have $A = 3q + r$ for some integers q and r with $0 \leq r < 2$. We compute

$$n^2 + 1 = (3q + r)^2 + 1$$
$$= 9q^2 + 6qr + r^2 + 1$$

Since ~~x~~ $3 \nmid n^2+1$, we have that $n^2+1 = 3t$ for some $t \in \mathbb{Z}$. It follows that $3t = 9q^2 + 6qr + r^2 + 1$, and so ~~$r^2+1$~~

$$r^2 + 1 = 3t - 9q^2 - 6qr = 3(t - 3q^2 - 2qr).$$

Therefore $3 \mid r^2+1$. But $0 \leq r < 2$ implies that $r^2 + 1 \in \{1, 2, 5\}$, and none of these is divisible by 3, a contradiction. ☒