Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[4 parts, 1 point each]** Suppose that the following sentences appear in a proof. If the sentence is stylistically poor or grammatically incorrect, then rewrite the sentence to fix these problems. Otherwise, write "OK".

   (a) The definition of an odd integer is $x = 2a + 1$ for any $a \in \mathbb{Z}$.

   By the definition of an odd integer, $x = 2a+1$ for some $a \in \mathbb{Z}$.

   Note: "By the definition of ...." is the proper way to invoke a definition, which is the intention here.

   (b) If one set is not a $\subseteq$ of another, then they are $\neq$.

   If one set is not a subset of another, then they are not equal

   Note: Use of notation $\subseteq$ and $\neq$ is improper here.

   (c) Each point in the plane is contained in infinitely many lines.

   OK

   (d) x is even $\Rightarrow x = 2k$ where $k \in \mathbb{Z}$.

   Since x is even, it follows that $x = 2k$ for some $k \in \mathbb{Z}$.

   OR, depending on context: If x is even, then $x = 2k$ for some $k \in \mathbb{Z}$.

2. **[2 points]** Prove that if n is an integer and $36 \nmid n^2$, then $2 \nmid n$ or $3 \nmid n$.

Pf: We show the contrapositive: If $2 \mid n$ and $3 \mid n$, then $36 \mid n^2$.

Suppose $2 \mid n$ and $3 \mid n$. There exist integers k_1 and k_2 such that $n = 2k_1$ and $n = 3k_2$. Since $n = 2k_1$, we have that n is even. Since $n = 3k_2$, 3 is odd, and an odd integer times an odd integer is odd, it must be that k_2 is even. Therefore $k_2 = 2k_3$ for some $k_3 \in \mathbb{Z}$.

We compute $n^2 = (3k_2)^2 = 9k_2^2 = 9(2k_3)^2 = 9 \cdot 4 \cdot k_3^2 = 36 k_3^2$.

Since $k_3^2 \in \mathbb{Z}$, it follows that $36 \mid n^2$. ∎

3. **[2 points]** Suppose that $x \in \mathbb{R}$. Prove that if $x^3 - 2x^2 - 3x \geq 0$, then $x \geq -1$.

Pf We show the contrapositive: If $x < -1$, then $x^3 - 2x^2 - 3x < 0$. ~~Note that~~ Suppose that $x < -1$, and note that

$$x^3 - 2x^2 - 3x = x(x^2 - 2x - 3)$$
$$= x(x-3)(x+1).$$

Since $x < -1$, all three factors, x, $x-3$, and $x+1$, are negative. The product of three negative real numbers is negative, and so $x^3 - 2x^2 - 3x = x(x-3)(x+1) < 0$. ▨

4. **[2 points]** Let $a \in \mathbb{Z}$. Show that $a^2 \equiv a \pmod 2$.

Pf. We give a direct proof with 2 cases, depending on the parity of a.

Case 1: Suppose that a is even, and so $a = 2k$ for some $k \in \mathbb{Z}$.

We compute $a^2 - a = (2k)^2 - (2k) = 4k^2 - 2k = 2(2k^2 - k)$

Since $2 \mid a^2 - a$, we have $a^2 \equiv a \pmod 2$ by definition.

Case 2: Suppose that a is odd, and so $a = 2k+1$ for some $k \in \mathbb{Z}$.

We compute $a^2 - a = (2k+1)^2 - (2k+1) = (4k^2 + 4k + 1) - (2k+1) = 4k^2 + 2k$.

Since $a^2 - a = 4k^2 + 2k = 2(2k^2 + k)$, we have that $2 \mid a^2 - a$. This means that $a^2 \equiv a \pmod 2$.

In both cases, $a^2 \equiv a \pmod 2$. ▨