Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[2.5 points]** Prove that if n is an odd integer, then $n^2 + 2n + 3$ is even.

Suppose that n is odd. We have that $n = 2a+1$ for some integer $a \in \mathbb{Z}$. We compute

$$\begin{aligned} n^2 + 2n + 3 &= (2a+1)^2 + 2(2a+1) + 3 \\ &= (4a^2 + 4a + 1) + (4a+2) + 3 \\ &= 4a^2 + 8a + 6 \\ &= 2(2a^2 + 4a + 3). \end{aligned}$$

Since $2a^2 + 4a + 3 \in \mathbb{Z}$, it follows that $n^2 + 2n + 3$ is even. $\blacksquare$

2. **[2.5 points]** Let d and n be integers. Prove that if $d \mid n$ and $d+1 \mid n$, then $d(d+1) \mid n$.

Suppose that $d \mid n$ and $d+1 \mid n$. There exist integers k_1 and k_2 such that $n = k_1 d$ and $n = k_2(d+1)$. Subtracting these two equations gives

$$0 = k_1 d - k_2(d+1) = k_1 d - k_2 d - k_2 = d(k_1 - k_2) - k_2,$$

and so $k_2 = d(k_1 - k_2)$. Substituting this expression for k_2 into $n = k_2(d+1)$ gives $n = [d(k_1 - k_2)](d+1) = (k_1 - k_2)(d(d+1))$. Since $k_1 - k_2 \in \mathbb{Z}$, it follows that $d(d+1) \mid n$. $\blacksquare$

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Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[2 parts, 2 points each]** *Proof critiques.* Give a critique of each claimed proof below. A *proof critique* addresses the following questions: (1) Is the proof correct? (2) If correct, can the proof be improved in some way? (3) If incorrect, what is/are the error(s)? Can they be fixed, and if so, how?

   (a) **Theorem 1.** *If n is a positive integer and 2^n is odd, then $2n$ is odd.*

   Proof: Let n be a positive integer. Note that 2^{n-1} is an integer since $n \geq 1$. Since $2^n = 2 \cdot 2^{n-1}$, it follows that 2^n is even. Since there are no integers which meet the conditions of the hypotheses, the desired conditional statement is true. □

   This proof is $\boxed{\text{correct}}$. The statement of the theorem is a bit odd, since it would be more natural and informative to state that 2^n is even whenever n is a positive integer. However, a conditional of the form "FALSE STATEMENT ⇒ FALSE STMT." is valid. (Note: the proof does assume that an even integer is not odd. This is not quite so obvious as it sounds from our definitions but we accept it without additional justification.)

   (b) **Theorem 2.** *Let a and b be integers. If $a \mid b$, then $|a| \leq |b|$.*

   Proof: Since $a \mid b$, we have that $b = ka$ for some integer k. Taking the absolute value of both sides gives $|b| = |ka| = |k| \cdot |a|$. Note that the product of two positive integers is at least as large as the factors. Therefore $|b| = |k| \cdot |a| \geq |a|$. □

   This proof is $\boxed{\text{incorrect}}$. There is a problem if k or a is zero. In that case, $|k| \cdot |a|$ is not the product of two positive integers, as implied by the proof.

   We can fix it by changing the theorem to state that "if $a \mid b$, then $|a| \leq |b|$ or $b = 0$." If $b \neq 0$, then it is the case that $k \neq 0$ and $a \neq 0$, and so the argument above works in this case.

   Note: "$b=0$" is a necessary addition to the conclusions; for example, if $a=5$ and $b=0$, then $5 \mid 0$ but it is false that $|5| \leq |0|$.