Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[5 points]** Recall the Fibonacci sequence, given by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Show that for $n \in \mathbb{N}$, we have $\sum_{k=1}^{n} F_k = F_{n+2} - 1$.

$F_2 = F_1 + F_0 = 1 + 0 = 1$

$F_3 = F_2 + F_1 = 1 + 1 = 2$

By induction on n.

Basis Step: Suppose $n = 1$. The LHS is $\sum_{k=1}^{1} F_k$, or F_1, which equals 1. The RHS is $F_{1+2} - 1$, or $F_3 - 1$, which equals 1 also.

Inductive Step: Suppose $n \geq 2$. We compute

$$\sum_{k=1}^{n} F_k = \left(\sum_{k=1}^{n-1} F_k \right) + F_n$$

$$= \left(F_{(n-1)+2} - 1 \right) + F_n \qquad \text{[Inductive Hypothesis]}$$

$$= F_{n+1} + F_n - 1$$

$$= F_{n+2} - 1$$

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2. **[5 points]** Prove that if n is an integer and $n \geq 16$, then there exist *positive* integers x and y such that $n = 3x + 5y$. (Hint: consider treating more than 1 case in the basis step.)

By induction on n.

Basis step: Suppose $16 \leq n \leq 18$. Note that

$$16 = 3 \cdot 2 + 5 \cdot 2$$

$$17 = 3 \cdot 4 + 5 \cdot \underline{1}$$

$$18 = 3 \cdot \underline{1} + 5 \cdot 3$$

Therefore the claim holds when $16 \leq n \leq 18$.

Inductive Step: Suppose $n \geq 19$. By the inductive hypothesis, ~~then~~ since $n-3 \geq 16$, there exist positive $x, y \in \mathbb{Z}$ such that $n-3 = 3x + 5y$. Since $n = 3(x+1) + 5y$, ~~it follows that~~ and $x+1$ and y are both positive integers, the claim follows. ∎