Math 283 **Quiz 11** **Fri. Apr. 13, 2018**

Name: Solutions

Directions: Solve the following problems. Give supporting work/justification where appropriate.

1. **[2 parts, 3 points each]** Prove or disprove the following.

   (a) Suppose that $a, b \in \mathbb{Z}$. If $a \mid b$ and $b \mid a$, then $a = b$.

This is false. For example, if $a = 3$ and $b = -3$, then $a \mid b$ (since $-3 = (-1)3$) and $b \mid a$ (since $3 = (-1)(-3)$), but it is not true that $a = b$.

   (b) If $n \in \mathbb{N}$, then $n^3 + 8$ is not prime.

$$
\begin{array}{r}
n^2 - 2n + 4 \\
n+2 \,\overline{)\, n^3 + 8} \\
\underline{n^3 + 2n^2} \\
-2n^2 + 8 \\
\underline{-2n^2 - 4n} \\
4n + 8 \\
\underline{4n + 8} \\
0
\end{array}
$$

This is true. Suppose $n \in \mathbb{N}$. Note that

$$n^3 + 8 = (n+2)(n^2 - 2n + 4).$$

Since $n \geq 1$, we have $3 \leq n+2 \leq n^3+2 < n^3 + 8$.

Since $n^3 + 8$ has a divisor between 3 and $(n^3+8)-1$,

it follows that $n^3 + 8$ is not prime.

2. **[4 points]** Prove that if $n \in \mathbb{N}$, then $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$.

Suppose for a contradiction that the identity is false and let n be a smallest counter-example. Since $\sum_{k=1}^{1} k(k+1) = 1 \cdot 2 = \frac{1(1+1)(1+2)}{3}$ it follows that $n \geq 2$. Note that $\sum_{k=1}^{n} k(k+1) = \left(\sum_{k=1}^{n-1} k(k+1)\right) + n(n+1)$

Since n is a minimum counter-example, the identity $(*)$ holds at $n-1$ and therefore $\sum_{k=1}^{n-1} k(k+1) = \frac{(n-1)n(n+1)}{3}$.

We have $\sum_{k=1}^{n} k(k+1) = \left(\sum_{k=1}^{n-1} k(k+1)\right) + n(n+1)$

$$= \frac{(n-1)n(n+1)}{3} + \frac{3n(n+1)}{3}$$

$$= \frac{n(n+1)\left[(n-1)+3\right]}{3}$$

$$= \frac{n(n+1)(n+2)}{3}.$$

Therefore n is not a counter-example, a contradiction. $\blacksquare$