Directions: You may work to solve these problems in groups, but all written work must be your own. Unless the problem indicates otherwise, all problems require some justification; a correct answer without supporting reasoning is not sufficient. Submissions must be stapled. See "Guidelines and advice" on the course webpage for more information.

1. *Proof critiques.* Give a critique of each claimed proof below. A *proof critique* addresses the following questions: (1) Is the proof correct? (2) If correct, can the proof be improved in some way? (3) If incorrect, what is/are the error(s)? Can they be fixed, and if so, how?

   (a) **Theorem 1.** *If x and y are real numbers, then $\frac{x+y}{2} \geq \sqrt{xy}$.*

   Proof:

   $$\begin{aligned} \frac{x+y}{2} &\geq \sqrt{xy} \\ x+y &\geq 2\sqrt{xy} \\ (x+y)^2 &\geq 4xy \\ x^2+2xy+y^2 &\geq 4xy \\ x^2-2xy+y^2 &\geq 0 \\ (x-y)^2 &\geq 0 \end{aligned}$$

   □

   (b) **Theorem 2.** *All real numbers are equal.*

   Proof: Let x and y be real numbers. Observe that

   $$x^2 - y^2 = (x-y)(x+y) = x(x+y) - y(x+y).$$

   After rearranging terms, this becomes $x^2 - x(x+y) = y^2 - y(x+y)$. Adding $\frac{(x+y)^2}{4}$ to both sides gives $x^2 - x(x+y) + \frac{(x+y)^2}{4} = y^2 - y(x+y) + \frac{(x+y)^2}{4}$. Factoring both sides, we see that $(x - \frac{x+y}{2})^2 = (y - \frac{x+y}{2})^2$ and taking the square root gives $x - \frac{x+y}{2} = y - \frac{x+y}{2}$. Adding $\frac{x+y}{2}$ to both sides gives $x = y$. Since x and y were arbitrarily chosen real numbers, it follows that all real numbers are equal. □

   (c) **Theorem 3.** *If $n \in \mathbb{Z}$, then $n^2 = 3k$ or $n^2 = 3k+1$ for some $k \in \mathbb{Z}$.*

   Proof: Suppose that $n \in \mathbb{Z}$. By the division algorithm, it follows that $n = 3q + r$ for some integers q and r with $0 \leq r < 3$. Since r is an integer and $0 \leq r < 3$, it follows that $r \in \{0, 1, 2\}$. We consider three cases, depending on the value of r.

   Case 1: If $r = 0$, then $n^2 = (3q+0)^2 = 9q^2 = 3(3q^2)$, and so $n^2 = 3k$ when we set k equal to the integer $3q^2$.

   Case 2: If $r = 1$, then $n^2 = (3q+1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1$, and so $n^2 = 3k + 1$ when we set k equal to the integer $3q^2 + 2q$.

   Case 3: If $r = 2$, then $n^2 = (3q+2)^2 = 9q^2 + 12q + 4 = 3(3q^2 + 4q + 1) + 1$, and so $n^2 = 3k + 1$ when we set k equal to the integer $3q^2 + 4q + 1$.

   In all cases, we have that $n^2 = 3k$ or $n^2 = 3k + 1$ for some integer k. □

   (d) **Theorem 4.** *Let $a, b, c \in \mathbb{Z}$. If $a \mid b$ and $b \mid c$, then $a \mid c$.*

Proof: Since $a \mid b$, we have that $b = ka$ for some integer k. Similarly, since $b \mid c$, it follows that $c = kb$ for some integer k. Therefore $c = kb = k(ka) = k^2a$. Since k^2 is an integer, it follows that $a \mid c$. □

2. Prove that if x is an odd integer, then x^3 is odd.

3. Prove that if x and y are integers and x is even, then xy is even.

4. Prove that if $n \in \mathbb{Z}$, then $5n^2 + 3n + 7$ is odd. Hint: try cases.

5. An integer p is *prime* if $p \geq 2$ and the only positive divisors of p are 1 and p. Prove that if n is a positive integer, $n \geq 2$, and n is not prime, then $2^n - 1$ is not prime.

6. *Efficient statements.* Using only the logical operands $\wedge$ and $\vee$, write a statement φ which is true if and only if at least two of $\{P_1, \ldots, P_8\}$ are true. Your sentence should use each P_j at most 3 times. For example,

$$[P_1 \wedge (P_2 \vee \ldots \vee P_8)] \vee [P_2 \wedge (P_3 \vee \ldots \vee P_8)] \vee \ldots \vee [P_7 \wedge P_8]$$

is logically equivalent to the desired sentence φ, but it is not efficient enough since P_8 is used a total of 7 times.