Directions: You may work to solve these problems in groups, but all written work must be your own. Unless the problem indicates otherwise, all problems require some justification; a correct answer without supporting reasoning is not sufficient. Submissions must be stapled. See "Guidelines and advice" on the course webpage for more information.

1. Suppose $B \neq \varnothing$ and $A \times B \subseteq B \times C$. Prove that $A \subseteq C$.

2. Each of the following is true or false. Decide which is the case and prove or disprove accordingly, using any method.

   (a) For every natural number n, the integer $n^2 + 17n + 17$ is prime.

   (b) If $a, b, c \in \mathbb{N}$ and ab, bc, and ac all have the same parity, then a, b, and c all have the same parity.

   (c) If A and B are finite sets, then $|A \cup B| = |A| + |B|$.

   (d) If A and B are sets, then $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

   (e) If A and B are sets, then $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$.

   (f) If p and q are prime numbers for which $p < q$, then $2p + q^2$ is odd.

3. Prove the following using induction or a "no smallest counterexample" argument.

   (a) For each $n \in \mathbb{N}$, we have $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

   (b) For each $n \in \mathbb{N}$, we have $\displaystyle\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$.

4. Suppose that $\alpha \in \mathbb{R}$ and $0 < \alpha < 1$. A magical cake has icing on one side. A baker cuts the cake to make a slice with center angle $\alpha \cdot 2\pi$ (radians), flips the slice over (so the piece has icing face-down), and the slice magically reattaches to the rest of the cake. The baker continues to make wedge slices with center angle $\alpha \cdot 2\pi$, proceeding counter-clockwise around the cake with each subsequent slice starting where the previous slice ended. For which $\alpha \in \mathbb{R}$ will this process eventually lead to the cake again having all its icing on the same side (up or down)? For which $\alpha \in \mathbb{R}$ will this process lead to the cake having all its icing facing down?