Directions: You may work to solve these problems in groups, but all written work must be your own. Unless the problem indicates otherwise, all problems require some justification; a correct answer without supporting reasoning is not sufficient. Submissions must be stapled. See "Guidelines and advice" on the course webpage for more information.

1. Prove the following (using any method).
   (a) If $a \in \mathbb{Z}$, then $a^3 \equiv a \pmod 3$.
   (b) If p is an integer, $p \geq 2$, and $n \nmid p$ for each integer n such that $2 \leq n \leq \sqrt{p}$, then p is prime.
   (c) Suppose $a, b, c \in \mathbb{Z}$. If $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$. Hint: use Proposition 7.1.
   (d) Suppose $a, b, p \in \mathbb{Z}$ and p is prime. If $p \mid ab$, then $p \mid a$ or $p \mid b$. Hint: use part (c).
2. *Proofs involving sets.* Prove the following.
   (a) $\{6n \colon n \in \mathbb{Z}\} = \{2n \colon n \in \mathbb{Z}\} \cap \{3n \colon n \in \mathbb{Z}\}$.
   (b) $\{9^n \colon n \in \mathbb{Q}\} = \{3^n \colon n \in \mathbb{Q}\}$
   (c) If A and B be sets, then $A \subseteq B$ if and only if $A \cap B = A$.
   (d) $\bigcap_{x \in \mathbb{R}} [3 - x^2, 5 + x^2] = [3, 5]$.
3. Prove that for all $d, n \in \mathbb{N}$, there exists a prime p such that $p \geq n$ and $p + d$ is not prime. (Hint: try proof by contradiction. What does it mean for this claim to be false? In other words, what is the logical negation of this claim? If you have trouble formulating the negation, then it may be helpful to translate the claim into a symbolic logic formula φ, negate φ and simplify, and finally translate back into English.)
4. Let n be an integer such that $n \geq 3$, and suppose that n lights are arranged in a circle. Initially, all lights are off. Each light is attached to a switch, but flipping a switch toggles the on/off status of its light and the two neighboring lights. In terms of n, what is the minimum number of switch flips needed to turn all lights on? Prove that your answer is correct.