Directions: Solve 5 of the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. *Hales–Jewett extension.* Let $\tau \in ([t] \cup \{\star_1, \ldots, \star_m\})^n$, where each $\star_j$ appears in at least one coordinate of τ. For $x \in [t]^m$, let $\tau(x) \in [t]^n$ be the vector obtained from τ by replacing each occurrence of $\star_j$ in τ with x_j. For example, with $\tau = 2\star_2 5\star_1 82\star_2 3$, we have $\tau(4,1) = 21548213$. The *combinatorial m-space rooted at* τ is $\{\tau(x)\colon\, x \in [t]^m\}$. Let $\mathrm{HJ}_m(r,t)$ be the minimum n such that every r-coloring of $[t]^n$ contains a monochromatic combinatorial m-space. Prove that $\mathrm{HJ}_m(r,t) \le m[\mathrm{HJ}(r,t^m)]$.

2. *Nearly spanning cycles.* Show that there exists a constant c such that for all positive α and ε with $\alpha > \varepsilon$, there exists n_0 such that the following holds for $n \ge n_0$. Every ε-regular pair with disjoint vertex sets X and Y of size n with density α has a cycle through all but at most $c \cdot \frac{\varepsilon}{\alpha-\varepsilon} n$ vertices.

3. *Sharpness example for Corrádi–Hajnal.* Prove that for every positive ε, there is an infinite family of graphs G such that $\delta(G) \ge (\frac{2}{3} - \varepsilon)|V(G)|$ but every subgraph of G with a triangle tiling has at most $(1-6\varepsilon)|V(G)|$ vertices.

4. *Tiling threshold for P_3.* Determine the least α such that if G is an n-vertex graph such that 3 divides n and $\delta(G) \ge \alpha n$, then G has a P_3-tiling. (Note: this requires a proof that $\delta(G) \ge \alpha n$ implies that G has a P_3-tiling, and also a construction of a sequence of graphs $G_1, G_2, \ldots$ such that G_k is a $3k$-vertex graph and $\delta(G) \ge (\alpha - o(1))3k$ but still G has no P_3-tiling.)

5. *Nearly spanning tiling threshold for P_3.*

   (a) Prove that for each $\varepsilon > 0$, there exists n_0 such that if G is an n-vertex graph with $n \ge n_0$ and $\delta(G) \ge \frac{1}{3}n$, then G has a P_3-tiling subgraph with at least $(1-\varepsilon)n$ vertices.

   (b) Give a family of examples that shows that the constant $1/3$ in part (b) is sharp.

6. An n-vertex graph with density ρ is (δ,γ)-*uniform* if $d(X,Y) \le (1+\delta)\rho|X||Y|$ when X and Y are disjoint vertex sets, each of size at least γn.

   Let r be an integer, and let ρ and δ be positive real numbers such that $r \ge 3$ and $\delta < \frac{1}{r-2}$. Show that there exists $\gamma > 0$ and n_0 such that if G is an n-vertex m-edge (δ,γ)-uniform graph with $n \ge n_0$ and $m \ge \rho\frac{n^2}{2}$, then $K_r \subseteq G$. (Hint 1: first try the case $r = 3$. Hint 2: let $\varepsilon = \varepsilon(r,\rho,\delta)$, and let $\alpha = \frac{1}{2r}\varepsilon^r$. With α chosen this way, obtain an α-regular partition and clean G with respect to a density threshold of 2ε. Then apply the embedding lemma with target graph K_r.)

   Comment: Turán's theorem states that a graph with density more than $1 - \frac{1}{r-1}$ contains a copy of K_r. In this exercise, we show that very small densities force a copy of K_r provided that the edges of G are distributed uniformly.