Directions: Solve 4 of the first 5 problems, plus problem number 6. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. For an integer m and a graph G, we use mG to denote the disjoint union of m copies of G. Prove that $R(mK_2, mK_2) = 3m - 1$.

2. Let S be a set of $R^{(3)}(m, m)$ points in the plane with no three on a line. Prove that S contains m points that form a convex m-gon. (Hint: assign to each triple $\{p, q, r\} \in \binom{S}{3}$ one of two colors that encodes appropriate information about their arrangement in the plane.)

3. Given graphs G_1 and G_2, the *induced Ramsey number*, denoted $R^\star(G_1, G_2)$, is the minimum number of vertices in a host graph H such that every red/blue edge-coloring of H contains an induced copy of G_1 that is all red or an induced copy of G_2 that is all blue.

   (a) Determine $R^\star(P_3, P_3)$, where P_3 denotes the path on 3 vertices.

   (b) Prove that if G is an n-vertex graph with m edges, then $R^\star(P_3, G) \le n + m$.

   Comment: it is not obvious that $R^\star(G_1, G_2)$ exists, since the requirement that our target graphs G_1 and G_2 be induced prevents using a large complete graph for H. The induced Ramsey theorem was discovered in the 1970's independently by several groups.

4. Use Behrend's construction to show that for some constant c and all sufficiently large n, there exists a subgraph G of $K_{n,n}$ with at least $n^{2-\frac{c}{\sqrt{\ln n}}}$ edges that is the union of at most $2n - 1$ induced matchings. (Hint: first obtain a relatively large subset of $[n]^2$ with no three points of the form $(x, y), (x + d, y), (x, y + d)$ with $d \neq 0$.)

5. *Turán Theorem stability for K_4-free graphs.* Let G be an n-vertex K_4-free graph with m edges. For each edge $e \in E(G)$, let $f(e)$ be the number of vertices that complete a triangle with e. Let t be the number of triangles in G, and let k be the number of copies of K_4^- in G. (Here, K_4^- denotes the graph obtained from K_4 by deleting an edge.)

   (a) Prove that $3t = \sum_{e \in E(G)} f(e)$ and $k = \sum_{e \in E(G)} \binom{f(e)}{2}$.

   (b) Prove that $2k \ge \frac{9t^2}{m} - 3t$.

   (c) Let B be the (X, Y)-bigraph where X is the set of triangles in G and $Y = |V(G)|$ with $T \in X$ and $y \in Y$ adjacent if and only if T and y together form a copy of K_4^- in G. Prove that there exists $T \in X$ such that T has at least $\frac{9t}{m} - 3$ neighbors.

   (d) Prove that for every $\varepsilon > 0$, there exists $\delta > 0$ such that every n-vertex K_4-free graph with at least $(\frac{1}{3} - \delta)n^2$ edges can be made 3-colorable by deleting at most εn vertices. (Hint: regularity is not needed. You may find our lower bound on t from HW1 useful.)

6. **[Required problem]** Prove that for each $\varepsilon > 0$, there exists a constant C such that if G is an n-vertex triangle-free graph with $\delta(G) \ge (\frac{1}{3} + \varepsilon)n$, then $\chi(G) \le C$. (Hint: choose an appropriate α in terms of ε and let $\{X_1, \ldots, X_M\}$ be an α-regular equipartition of $V(G)$. Show that for each vertex $v \in V(G)$, there is a part X_i such that v has more than $\frac{1}{2}|X_i|$ neighbors in X_i.)