

Directions: Solve 5 of the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. For $r \geq 3$, determine the minimum number of vertices in a graph G such that $\chi(G) \geq r$ but G contains no copy of K_r .
2. Let S be a set of n points in the plane such that $|p - q| < 1$ when $p, q \in S$. Show that at most $\lfloor n^2/3 \rfloor$ pairs of points in S are at distance at least $1/\sqrt{2}$.
3. Let G be a copy of K_n in which every edge is colored red or blue. For $v \in V(G)$, let r_v be the number of red edges incident to v and let b_v be the number of blue edges incident to v .
 - (a) Find a formula for the number of monochromatic triangles in G in terms of n , the red degrees r_v , and the blue degrees b_v . (Hint: first count the number of triangles that are not monochromatic.)
 - (b) Prove that G has at least $n(n-1)(n-5)/24$ monochromatic triangles.
 - (c) Describe the 2-edge-colorings of K_n that have exactly $n(n-1)(n-5)/24$ monochromatic triangles.
4. Let G be a graph and let $X, Y \subseteq V(G)$. Prove that for each $\varepsilon > 0$, there exists $\delta > 0$ such that if (X, Y) is δ -regular, then (X', Y') is ε -regular when $X' \subseteq X$, $Y' \subseteq Y$, $|X'| \geq \varepsilon|X|$, and $|Y'| \geq \varepsilon|Y|$.
5. An *equipartition* is a partition whose parts differ in size by at most 1.
 - (a) Show that for each $\delta > 0$, there exists ℓ such that if X is a partition of $V(G)$ into k parts, then there is an equipartition Y of $V(G)$ into ℓk parts such that $\text{msd}(Y) \geq \text{msd}(X) - \delta$.
 - (b) Describe how to modify the proof of Szemerédi's regularity lemma to obtain the following strengthening: for each $\varepsilon > 0$, there exists M such that every graph has an ε -regular equipartition into at most M parts.
6. Prove that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if G is an n -vertex graph with $|E(G)| \geq (\frac{1}{4} + \varepsilon)\frac{n^2}{2}$, then either G contains a copy of K_4 , or G has an independent set of size at least δn .