Directions: Solve the following problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. [8.1] Let p be a prime and let $d = \gcd(m, p-1)$. Prove that $N(x^m = a) = \sum_{\{\chi:\ \chi^d = \varepsilon\}} \chi(a)$.

2. [8.{3,4,6}] Let χ be a nontrivial character of F_p and let ρ be the character of order 2.
   (a) Show that $\sum_t \chi(1-t^2) = J(\chi, \rho)$. (Hint: evaluate $J(\chi, \rho)$ using the relation $N(x^2 = a) = 1 + \rho(a)$.)
   (b) Let $k \in F_p^\star$. Show that $\sum_t \chi(t(k-t)) = \chi(k^2/2^2)J(\chi, \rho)$. (Hint: multiply the identity in part (a) by $\chi(k^2/2^2)$ and use a change of variable.)
   (c) Show that $J(\chi, \chi) = \chi(2)^{-2} J(\chi, \rho)$. (Hint: Apply (b) with $k = 1$.)

3. [8.7] Suppose that $p \equiv 1 \pmod 4$ and that χ is a character of order 4, and let $\rho = \chi^2$. Prove that $J(\chi, \chi) = \chi(-1)J(\chi, \rho)$.

4. [8.{12,13}] Uniqueness of representations.
   (a) Suppose that $p \equiv 1 \pmod 4$, so that $p = a^2 + b^2$ for some $a, b \in \mathbb{Z}$. Prove that if we require a and b to be positive, with a odd and b even, then a and b are uniquely determined. (Hint: argue that $a + bi$ is prime in the unique factorization domain $\mathbb{Z}[i]$.)
   (b) Suppose that $p \equiv 1 \pmod 3$, so that $4p = A^2 + 27B^2$ for some $A, B \in \mathbb{Z}$. Prove that if we require that $A \equiv 1 \pmod 3$, then A is uniquely determined. (Hint: show that if $\alpha \in \mathbb{Z}[\omega]$ and $|\alpha|^2 = p$, then α is prime in the unique factorization domain $\mathbb{Z}[\omega]$.)

5. [8.14] Suppose that $p \equiv 1 \pmod n$ and that χ is a character of order n. Show that $g(\chi)^n \in \mathbb{Z}[\eta]$ where $\eta = e^{2\pi i/n}$.

6. [8.19] Find a formula for the number of solutions to $x_1^2 + \cdots + x_r^2 = 0$ in F_p. Hint: first show that the number of solutions is given by $p^{r-1} + J_0(\rho, \ldots, \rho)$, where ρ is the Legendre symbol (i.e. character of order 2) and there are r arguments to J_0. Then use Proposition 8.5.1 and Theorem 3.)

7. [9.{12,13,14}] Let $\omega = e^{2\pi i/3}$, $\lambda = 1 - \omega$, and $D = \mathbb{Z}[\omega]$.
   (a) Show that $\omega\lambda$ has order 8 in $D/5D$ and that $\omega^2\lambda$ has order 24. [Hint: first show that $(\omega\lambda)^2$ has order 4.]
   (b) Show that π is a cube in $D/5D$ if and only if π is congruent modulo 5 to an element in $\{1, 2, 3, 4, 1 + 2\omega, 2 + 4\omega, 3 + \omega, 4 + 3\omega\}$.
   (c) For which primes $\pi \in D$ is $x^3 \equiv 5 \pmod \pi$ solvable?

8. [9.15] Suppose that $p \equiv 1 \pmod 3$ and that $p = \pi\overline{\pi}$, where π is a primary prime in D. Let a be an integer. Show that $x^3 \equiv a \pmod p$ has an integer solution x if and only if $\chi_\pi(a) = 1$. (Hint: first argue that π and $\overline{\pi}$ are relatively prime. Be careful to obtain an integer solution $x \in \mathbb{Z}$, and not just a solution $x \in D$; it may help to recall that $\{0, 1, \ldots, p-1\}$ is a complete set of representatives for $D/\pi D$.)