Directions: Solve the following problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. [IR 1.10] Suppose that $(u, v) = 1$. Show that $(u + v, u - v)$ is either 1 or 2.
2. [IR 1.{15,18}]
   (a) Prove that a positive integer a is the square of another integer if and only if $\operatorname{ord}_p a$ is even for all primes p. Give a generalization.
   (b) Let m be a positive integer. Prove that $m^{1/n}$ is irrational if m is not the nth power of an integer. (In other words, prove that there do not exist integers a and b such that $m^{1/n} = \frac{a}{b}$ if m is not the nth power of an integer.)
3. [IR 1.21] Prove that $\operatorname{ord}_p(a + b) \geq \min(\operatorname{ord}_p a, \operatorname{ord}_p b)$ with equality when $\operatorname{ord}_p a \neq \operatorname{ord}_p b$.
4. [IR 1.{24,26}]
   (a) Prove the following.
      i. $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-1})$
      ii. If n is odd, then $x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 \cdots + y^{n-1})$.
   (b) Let a and n be positive integers. Prove that if $a \geq 2$ and $a^n + 1$ is a prime, then a is even and n is a power of two. (Primes of the form $2^{2^t} + 1$ are called Fermat primes. It is not known if there are infinitely many primes.)
5. [IR 1.34] Show that 3 is divisible by $(1 - \omega)^2$ in $\mathbb{Z}[\omega]$.
6. [IR 1.{33,38}]
   (a) Show that $\alpha \in \mathbb{Z}[i]$ is a unit if and only if $\lambda(\alpha) = 1$. Deduce that the set of units of $\mathbb{Z}[i]$ is $\{1, -1, i, -i\}$.
   (b) Suppose that $\pi \in \mathbb{Z}[i]$ and that $\lambda(\pi)$ is a prime in the integers. Show that π is prime in $\mathbb{Z}[i]$.
7. [IR 1.39] Show that in any integral domain, a prime element is irreducible.