Name: ____________________

Directions: Solve the following problems. Most answers require explanation in English sentences.

1. **[4 parts, 2.5 points each]** Evaluate the following sums. (No explanation necessary.)

   (a) $\displaystyle\sum_{j=1}^{3n} j$

   (b) $\displaystyle\sum_{i=1}^{2n} i+1$

   (c) $\displaystyle\sum_{j=1}^{n}\left(\sum_{i=1}^{j} 1\right)$

   (d) The sum of all integers in $\{1,\ldots,400\}$ that are not divisible by 3. The sum begins $1+2+4+5+7+8+10+\ldots$.

2. **[3 points]** Let $A = \{(1,2), 3, 3, 3, 3, \{4,5\}\}$, $B = \{(2,1), 3, 3, 3, 3, \{5,4\}\}$, and $C = \{(a,b)\}$.

   (a) What is $A \cap B$?

   (b) What is $A \times C$?

   (c) What is $\mathbb{Z} \cap \mathcal{P}(\mathbb{Z})$?

3. Let P be the set of primes.

   (a) **[3 points]** Let E be the set of even integers that are at least 4. Give another definition of E using set builder notation.

   (b) **[2 points]** Let A be the set of all integers that can be expressed as the sum of two primes. Give another definition of A using set builder notation; you may use the set P.

   (c) **[2 points]** Using set theory notation and the sets E and A, state Goldbach's conjecture.

4. **[5 points]** Why do we need the axioms of set theory to impose restrictions on how sets may be defined? Give details.

5. **[5 points]** Let U be a universe, and let A and B be subsets of U. Prove that if $A \subseteq B$, then $\overline{B} \subseteq \overline{A}$.

6. For each $n \in \mathbb{N}$, let $A_n = \{x \in \mathbb{R} \colon 1 - \frac{1}{n} < x \leq n\}$.

   (a) **[4 points]** What is A_1, A_2, and A_3?

   (b) **[3 points]** What is $\displaystyle\bigcap_{n \in \mathbb{N}} A_n$?

   (c) **[3 points]** What is $\displaystyle\bigcup_{n \in \mathbb{N}} A_n$?

7. [**4 parts, 2.5 points each**] Translate the following into formal, symbolic logic, defining any sets and variable propositions needed. Then, decide whether the statement is True or False.

   (a) Every odd natural number is prime.

   (b) Strictly between any two distinct real numbers, lies a rational number.

   (c) Some quadratic polynomial with integer coefficients has no real roots.

   (d) There exists an integer that can be written as the sum of two primes in two different ways.

8. **[2 parts, 3 points each]** Let $P(x)$ be "$\exists k \in \mathbb{Z}.\, x = 2k$" and let $Q(x, y)$ be "$x^2 + y^2 = 1$". Translate the following into English. Then, decide whether the statement is True or False.

   (a) $\forall x \in \mathbb{Z}.\, \forall y \in \mathbb{Z}.\, P(x) \wedge \neg P(y) \implies \neg P(x + y)$.

   (b) $\exists x \in \mathbb{R}.\, \exists y \in \mathbb{R}.\, (P(x) \vee P(y)) \wedge (Q(x, y))$.

9. **[4 points]** Let A be the set of primes, and let $Q(x)$ be the property that every integer divisor of x is strictly larger than $\sqrt{x}$. Let P be the statement "$\forall x \in \mathbb{N}.\, Q(x) \implies x \in A$". Express $\neg P$ using formal logical symbols. Then, decide which one of $\{P, \neg P\}$ is True.