Name: Solutions

Directions: Solve the following problems. Most answers require explanation in English sentences.

1. **[4 parts, 2.5 points each]** Evaluate the following sums. (No explanation necessary.)

(a) $\sum_{j=1}^{3n} j$

Substitute $3n$ for n in $\frac{n(n+1)}{2}$:

$$\boxed{\frac{3n(3n+1)}{2}}$$

(b) $\sum_{i=1}^{2n} i+1$

$$= \left(\sum_{i=1}^{2n} i\right) + \left(\sum_{j=1}^{2n} 1\right) = \frac{2n(2n+1)}{2} + 2n$$

$$= 2n^2 + n + 2n$$

$$= \boxed{2n^2 + 3n} = \boxed{n(2n+3)}$$

(c) $\sum_{j=1}^{n}\left(\sum_{i=1}^{j} 1\right)$

$$= \sum_{j=1}^{n} (j) = \boxed{\frac{n(n+1)}{2}}$$

(d) The sum of all integers in $\{1, \ldots, 400\}$ that are not divisible by 3. The sum begins $1+2+4+5+7+8+10+\ldots$.

$$= \left(\sum_{i=1}^{400} i\right) - \left(\sum_{j=1}^{133} 3j\right) = \frac{400(401)}{2} - 3\sum_{j=1}^{133} j$$

$$= (200)(401) - 3\cdot\frac{(133)(134)}{2} = 80{,}200 - 3\cdot 67\cdot 133$$

$$= 80{,}200 - 26{,}733$$

$$= \boxed{53{,}467}$$

133
× 67
931
7980

8911
× 3
26733

2. [**3 points**] Let $A = \{(1,2), 3, 3, 3, 3, \{4,5\}\}$, $B = \{(2,1), 3, 3, 3, 3, \{5,4\}\}$, and $C = \{(a,b)\}$.

(a) What is $A \cap B$?

$A \cap B = \{3, \{4,5\}\}$

(b) What is $A \times C$?

$A \times C = \{ ((1,2), (a,b)), (3, (a,b)), (\{4,5\}, (a,b))\}$

(c) What is $\mathbb{Z} \cap \mathcal{P}(\mathbb{Z})$?

The elements of $\mathbb{Z}$ are integers.
The elts of $\mathcal{P}(\mathbb{Z})$ are sets.

$\mathbb{Z} \cap \mathcal{P}(\mathbb{Z}) = \emptyset$.

3. Let P be the set of primes.

(a) [**3 points**] Let E be the set of even integers that are at least 4. Give another definition of E using set builder notation.

$E = \{ n \in \mathbb{Z} : n \geq 4$ and $n = 2k$ for some $k \in \mathbb{Z} \}$

(b) [**2 points**] Let A be the set of all integers that can be expressed as the sum of two primes. Give another definition of A using set builder notation; you may use the set P.

Let P be the set of primes.

$A = \{ n \in \mathbb{Z} : \exists a, b \in P.\ n = a + b \}$

(c) [**2 points**] Using set theory notation and the sets E and A, state Goldbach's conjecture.

$E \subseteq A$

4. **[5 points]** Why do we need the axioms of set theory to impose restrictions on how sets may be defined? Give details.

Otherwise, set theory is inconsistent, as demonstrated by Russell's paradox. Let $R = \{A : A \notin A\}$. Now both $R \in R$ and $R \notin R$ are impossible; we need axioms that forbid us from defining such sets.

5. **[5 points]** Let U be a universe, and let A and B be subsets of U. Prove that if $A \subseteq B$, then $\overline{B} \subseteq \overline{A}$.

Suppose that $A \subseteq B$. We show that $\overline{B} \subseteq \overline{A}$.

Let $x \in \overline{B}$. It follows that $x \notin B$. We claim that $x \notin A$ also. Indeed, if $x \in A$, ~~were~~ then since $A \subseteq B$ it would follow that $x \in B$, which contradicts $x \notin B$. The contradiction implies $x \notin A$. Since $x \notin A$, we have $x \in \overline{A}$.

Since every element in $\overline{B}$ is also in $\overline{A}$, we have that $\overline{B} \subseteq \overline{A}$. ▨

6. For each $n \in \mathbb{N}$, let $A_n = \{x \in \mathbb{R} : 1 - \frac{1}{n} < x \leq n\}$.

(a) [**4 points**] What is A_1, A_2, and A_3?

$A_1 = \{x \in \mathbb{R} : 0 < x \leq 1\} = (0, 1]$

$A_2 = \{x \in \mathbb{R} : \frac{1}{2} < x \leq 2\} = (\frac{1}{2}, 2]$

$A_3 = \{x \in \mathbb{R} : \frac{2}{3} < x \leq 3\} = (\frac{2}{3}, 3]$

(b) [**3 points**] What is $\bigcap_{n \in \mathbb{N}} A_n$?

We claim $\boxed{\bigcap_{n \in \mathbb{N}} A_n = \{1\}.}$ For each $n \in \mathbb{N}$, we have $1 - \frac{1}{n} < 1 \leq n$, and so $1 \in \bigcap_{n \in \mathbb{N}} A_n$. Since every elt. in $\bigcap_{n \in N} A_n$ belongs to ~~A_1 area~~ $(0, 1]$, and eventually A_n excludes x when $0 < x < 1$, it follows that $\bigcap_{n \in N} A_n \subseteq \{1\}$.

(c) [**3 points**] What is $\bigcup_{n \in \mathbb{N}} A_n$?

We claim $\boxed{\bigcup_{n \in \mathbb{N}} A_n = \{x \in \mathbb{R} : x > 0\}.}$ Each A_n is a set of positive real numbers, so $\bigcup_{n \in \mathbb{N}} A_n \subseteq \{x \in \mathbb{R} : x > 0\}$.

If $0 < x < 1$, then $x \in A_1$. If $x \geq 1$, then $x \in A_n$ ~~where~~ for any integer n such that $n \geq x$. It follows that $\{x \in \mathbb{R} : x > 0\} \subseteq \bigcup_{n \in \mathbb{N}} A_n$.

7. **[4 parts, 2.5 points each]** Translate the following into formal, symbolic logic, defining any sets and variable propositions needed. Then, decide whether the statement is True or False.

(a) Every odd natural number is prime.

Let P be the set of primes and let O be the set of odd integers.

$\forall n \in O.\ \ n \in P.$

This is false; for example, $9 \in O$ but $9 \notin P$.

(b) Strictly between any two distinct real numbers, lies a rational number.

$\forall a, b \in \mathbb{R}.\ \ (a < b) \Rightarrow \left(\exists q \in \mathbb{Q}.\ \ a < q < b\right).$

This is true.

(c) Some quadratic polynomial with integer coefficients has no real roots.

$\exists a, b, c \in \mathbb{Z}.\ \ \forall x \in \mathbb{R}.\ \ ax^2 + bx + c \neq 0$

This is true; for example, $x^2 + 1$ has no real roots.

(d) There exists an integer that can be written as the sum of two primes in two different ways.

Let P be the set of primes.

$\exists n \in \mathbb{Z}.\ \ \exists a, b, c, d \in P.\ \ (n = a + b) \wedge (n = c + d) \wedge \{a, b\} \neq \{c, d\}$

This is true. For example, $10 = 3 + 7$ and $10 = 5 + 5$.

8. **[2 parts, 3 points each]** Let $P(x)$ be "$\exists k \in \mathbb{Z}.\, x = 2k$" and let $Q(x,y)$ be "$x^2 + y^2 = 1$". Translate the following into English. Then, decide whether the statement is True or False.

   (a) $\forall x \in \mathbb{Z}.\, \forall y \in \mathbb{Z}.\, P(x) \wedge \neg P(y) \implies \neg P(x+y)$.

   Note: $P(x)$ means x is an even integer.

   The sum of ~~odd~~ an even integer and an odd integer is odd.

   This is $\boxed{\text{true}}$

   (b) $\exists x \in \mathbb{R}.\, \exists y \in \mathbb{R}.\, (P(x) \vee P(y)) \wedge (Q(x,y))$.

   There is a real solution to $x^2+y^2=1$ in which at least one of $\{x, y\}$ is an even integer.

   This is $\boxed{\text{true}}$ for example $x=0$ and $y=1$ is such a solution.

9. **[4 points]** Let A be the set of primes, and let $Q(x)$ be the property that every integer divisor of x is strictly larger than $\sqrt{x}$. Let P be the statement "$\forall x \in \mathbb{N}.\, Q(x) \implies x \in A$". Express $\neg P$ using formal logical symbols. Then, decide which one of $\{P, \neg P\}$ is True.

   $\neg P$ is "$\exists x \in \mathbb{N}.\ Q(x) \wedge x \notin A$".

   In this case, P is $\boxed{\text{true}}$. If $x=1$, then $Q(x)$ fails. If $x \geq 2$, then either x is prime or x has a divisor less than $\sqrt{x}$.