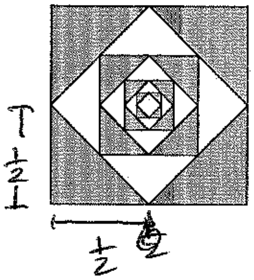
Name: Solutions.

Directions: All questions require explanation in English sentences.

1. **[10 points]** The midpoints of the sides of a square are joined to form another square, and this process is repeated. The outer square has side length 1. What is the total area of the shaded regions?

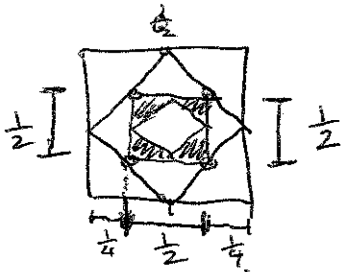
Let A be the total area.



Let A_1 be the area between the outermost square and the next square. We have $A_1 = 4\cdot\left(\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\right) = \frac{1}{2}$, since each of the 4 triangular regions has area $\frac{1}{8}$.

Let A_2 be the area of the remaining shaded regions.

We have that A_2 is a scaled down version of the total area A, where the scaling factor is $\frac{1}{2}$.



So $A_2 = \left(\frac{1}{2}\right)^2 A = \frac{1}{4}A$. From $A = A_1 + A_2 = \frac{1}{2} + \frac{1}{4}A$, we get $\frac{3}{4}A = \frac{1}{2}$, so $A = \boxed{\frac{2}{3}}$.

2. **[2 parts, 5 points each]** Consider the following argument.

> **Theorem 1.** *If n is an integer and $n \geq 5$, then $n^2 - 16$ is not prime.*
>
> **Proof:** Using algebra, we see that $n^2 - 16 = (n+4)(n-4)$. Since $n+4$ divides $n^2 - 16$, we conclude that $n^2 - 16$ is not prime. □

(a) Execute the proof firstly for $n = 5$ and secondly for $n = 6$.

For $n=5$: the first assertion is $25-16 = 9 \cdot 1$; next, we claim $n+4$, or 9, divides n^2-16, or 9. This is true, but does not imply that 9 is not prime.

For $n=6$: We have $6^2-16 = (6+4)(6-4)$, or $20 = 10 \cdot 2$. Next, we claim that since 10 divides 20, it follows that 20 is not prime.

(b) Analyze the proof above. Is it a valid proof? If not, can it be corrected? If possible, how would you correct it?

No, there is an error when $n=5$. This can be corrected by analyzing the case $n=5$ separately and noting that for $n \geq 6$, we have that

$$n^2 - 16 = (n+4)(n-4)$$

and both $n-4 \geq 2$ and $n+4 \geq 2$. Therefore, for $n \geq 6$, the factorization $n^2-16 = (n+4)(n-4)$ does prove that n^2-16 is not prime.

3. **[2 parts, 5 points each]** Consider the following argument.

Theorem 2. *If a and b are nonnegative real numbers, then $(a+b)/2 \geq \sqrt{ab}$.*

Proof: Since the square of a real number is nonnegative, we have $(a-b)^2 \geq 0$. Expanding the left hand side, we obtain $a^2 - 2ab + b^2 \geq 0$. Adding $4ab$ to both sides, we see that $a^2+2ab+b^2 \geq 4ab$, or $(a+b)^2 \geq (2\sqrt{ab})^2$. Since $a+b \geq 0$ and $2\sqrt{ab} \geq 0$, we may take the square root of both sides, obtaining $a+b \geq 2\sqrt{ab}$. Dividing both sides by 2, we conclude $(a+b)/2 \geq \sqrt{ab}$. □

(a) Execute the proof for $a = 3$ and $b = 5$.

First, we have $(3-5)^2 \geq 2$, and so $3^2 - 2\cdot3\cdot5 + 5^2 \geq 0$.

Next, we add $4\cdot3\cdot5$ to both sides to get

$$3^2 + 2\cdot3\cdot5 + 5^2 \geq 4\cdot3\cdot5$$

or

$$(3+5)^2 \geq (2\sqrt{3\cdot5})^2.$$

Next, $(3+5) \geq 2\sqrt{3\cdot5}$, so $\frac{3+5}{2} \geq \sqrt{3\cdot5}$.

(b) Analyze the proof above. Is it a valid proof? If not, can it be corrected? If possible, how would you correct it?

Yes, this proof is correct. Most manipulations are clear. When A and B are ~~positive~~ non-negative, it is indeed the case that

$$A \geq B \text{ iff } A^2 \geq B^2.$$

4. [**5 points**] One of the following implications is true and the other is false. Identify which is which. Prove the true implication and find a counterexample for the other. Let a be a real number.

   - If a^2 is irrational, then a is irrational. True
   - If a is irrational, then a^2 is irrational. FALSE.

Note: If $a = \sqrt{2}$, then a is irrational but $a^2 = 2$ so a^2 is rational. So $a = \sqrt{2}$ is a counterexample to the second implication.

Thm. If a^2 is irrational, then a is irrational.

Pf: ~~Let a be a rea~~ ~~Suppose a^2 is irrational~~ Let a be a real number such that a^2 is irrational. Suppose for a contradiction that a is rational. Then $a = \frac{p}{q}$ for some integers p and q and $a^2 = \frac{p^2}{q^2}$. So a^2 is also rational, contradicting that a^2 is irrational.

5. [**5 points**] For which real values of a is the polynomial $x + a$ a factor of $x^3 + 3ax^2 - a$?

We have $x+a$ is a factor of x^3+3ax^2-a iff $-a$ is a root of x^3+3ax^2-a.

We solve
$$(-a)^3 + 3a(-a)^2 - a = 0$$
$$-a^3 + 3a^3 - a = 0$$
$$2a^3 - a = 0$$
$$2a\left(a^2 - \tfrac{1}{2}\right) = 0$$
$$2a\left(a - \tfrac{1}{\sqrt{2}}\right)\left(a + \tfrac{1}{\sqrt{2}}\right) = 0$$

So $x+a$ is a factor of x^3+3ax^2-a if and only if $a = 0$ or $a = -\frac{1}{\sqrt{2}}$ or $a = \frac{1}{\sqrt{2}}$.

6. [**4 parts, 2.5 points each**] Let (*) be the equation $3x^2 + (x-1)y = 4$. Decide whether the following statements are true or false. Explain your answer.

(a) For each real number x and each real number y, the pair x, y satisfies (*).

FAlse. For example when $x = y = 0$, the (*) is not satisfied.

(b) There exists a real number x such that for each real number y, the pair x, y satisfies (*).

FALSE. If $x=1$, then (*) becomes $3=4$, which no value of y satisfies. If $x \neq 1$, then (*) is equivalent to

$$y = \frac{4-3x^2}{x-1}$$ which is satisfied by exactly 1 value of y

(c) For each real number x, there exists a real number y such that the pair x, y satisfies (*).

FALSE. As noted in (b), when $x=1$, there is no value of y which satisfies (*).

(d) For each real number y, there exists a real number x such that the pair x, y is satisfies (*).

True. For a fixed real number y, the eqn (*) becomes the quadratic $3x^2 + yx - (y+4) = 0$. This has real solns iff the discriminant $y^2 - 4 \cdot 3[-(y+4)]$ is non-negative. Since $y^2 + 12y + 48 = (y+6)^2 + 12 > 0$, we conclude that for each value of y, there are two real values of x that satisfy (*).

7. **[10 points]** Let f and g be polynomials of degree at most n, and suppose that $a_1, \ldots, a_{n+1}$ are distinct real numbers such that $f(a_i) = g(a_i)$ for each i. Prove that $f = g$. Hint: let $h(x) = f(x) - g(x)$. What can you say about the degree of h?

Let $h(x) = f(x) - g(x)$. Either $h(x) = 0$ or $h(x)$ has degree at most n. For each i, we have

$$h(a_i) = f(a_i) - g(a_i) = 0$$

and therefore $a_1, \ldots, a_{n+1}$ are $n+1$ distinct roots of $h(x)$. Since ~~h(x) has~~ a polynomial of degree at most n has at most n distinct roots, it must be that $h(x) = 0$. Therefore $f(x) - g(x) = 0$, and so $f(x) = g(x)$. $\square$