Directions: You may work to solve these problems in groups, but all written work must be your own. **Show your work**; See "Guidelines and advice" on the course webpage for more information.

1. [S] Let a, b, and c be positive integers satisfying $a^2 + b^2 = c^2$. Is there a right triangle with side lengths a and b, and hypotenuse c? If so, how would you go about constructing it? If not, why not?

2. [S 1.5.2] A government mint is commissioned to produce gold coins. The mint has 20 machines, each of which is producing coins weighing 5 grams apiece. One day, the foreman of the mint notices that some coins are light, and after assessing the machines, he finds that one of them is making 4 gram coins, while the other 19 machines are working perfectly. He decides to use the situation to his advantage and identify the smartest employee, to be promoted next. He tells the workers that exactly one machine is producing coins that are 4 grams, and that they need to determine which machine is broken. You, as an employee, are allowed to take one, and only one, reading on a scale. You can place any number of coins from any machines, of your choosing, but you must pool them together and will only see the total weight of all the coins, as a number in grams. How do you do this so that you can determine precisely which machine is the broken one?

3. [S 1.5.14] Evaluate the following mathematical argument. Is it clearly written? Does it use correct logic? If not, where is the problem? Can it be fixed?

   Theorem 1. *Let a, b, c be real numbers, with $a \neq 0$. Then $-\frac{b}{2a}$ is a solution to the equation $ax^2 + bx + c = 0$.*

   Proof: Let x and y be solutions to the equation. Subtracting $ay^2 + by + c = 0$ from $ax^2 + bx + c = 0$ yields $a(x + y)(x - y) + b(x - y) = 0$. Hence, $a(x + y) + b = 0$, and so $x + y = -\frac{b}{a}$. Since x and y were any solutions, we may choose x and y so that $x = y$, in which case $2x = -\frac{b}{a}$, and therefore $x = -\frac{b}{2a}$ is a solution. $\square$

4. Let (*) be the equation $2x^2 + (x - 2)y = 8$. For each statement below, decide whether the statement is true or false and prove that your answer is correct.

   (a) There exist real numbers x and y such that the pair x, y satisfies (*).

   (b) For each real number x, there is a real number y such that the pair x, y satisfies (*).

   (c) For each real number y, there is a real number x such that the pair x, y satisfies (*).

   (d) There exists a real number x such that for every real number y, the pair x, y satisfies (*).

   (e) There exists a real number y such that for every real number x, the pair x, y satisfies (*).

   (f) For each real number x and each real number y, the pair x, y satisfies (*).