Directions: You may work to solve these problems in groups, but all written work must be your own. **Show your work**; See "Guidelines and advice" on the course webpage for more information.

1. Prove that $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$.

2. Suppose that a is a real number such that $a + (1/a)$ is an integer. Prove that $a^n + (1/a)^n$ is an integer for every nonnegative integer n. (Hint: compute $(a^{n-1} + (1/a)^{n-1})(a + (1/a))$.)

3. A strange country issues only two types of coins: one worth 7 cents and another worth 10 cents. Prove that if n is an integer and $n \geq 54$, then there is some combination of 7 and 10 cent coins that give n cents exactly. (Hint: find two solutions to $7x + 10y = 1$: one in which $x > 0$ and $y < 0$ and another in which $x < 0$ and $y > 0$.)

4. Suppose that 100 pebbles are placed in a pile. Two players take turns removing pebbles from the pile. In each turn, a player is allowed to remove 1, 2, 3, 4, or 5 pebbles from the pile. The last player to move wins. Prove that the first player has a winning strategy.

5. Suppose that n coins are arranged in a row. Coins may be heads up (H) or tails up (T). A coin that is heads up can be removed and any two remaining neighbors are flipped. When a coin between two other coins is removed, the removed coins do not become neighbors. For example, the following sequence is legal: "`THHTH, H.TTH, ..TTH, ..TH., ..H.., .....`"

   For which arrangements of heads and tails can we remove all the coins? As we have seen, `THHTH` is one such arrangement but one may check that `TTHTH` is not. Prove that your answer is correct. (Hint: for small n, write down explicitly which arrangements work and which do not. Do you see a pattern? Can you prove the pattern is correct?)