Directions: You may work to solve these problems in groups, but all written work must be your own. **Show your work**; See "Guidelines and advice" on the course webpage for more information.

1. Prove that $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ for each $n \in \mathbb{N}$.
2. Recall the Fibonacci sequence defined by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Prove that $f_{n-1}f_{n+1} = f_n^2 + (-1)^n$ for each $n \in \mathbb{N}$.
3. Suppose that $x_1, \ldots, x_n$ are real numbers in the interval $[0, 1]$. Prove that $\prod_{i=1}^{n}(1 - x_i) \geq 1 - \sum_{i=1}^{n} x_i$.
4. Prove that each positive integer n can be written as a sum of powers of 3 in which each power is used at most twice. For example, $306 = 3^5 + 3^3 + 3^3 + 3^2$.
5. Let $A = \{(x, y) \in \mathbb{Z}^2 \colon 0 \leq x \leq 40 \text{ and } 0 \leq y \leq 80\}$. Staring from position $(0, 0)$, two players alternate turns. In each turn, a player increases one of the coordinates and leaves the other unchanged to obtain a new position in A. The player who reaches position $(40, 80)$ first wins. Prove that the first player has a strategy to win no matter how the second player moves.