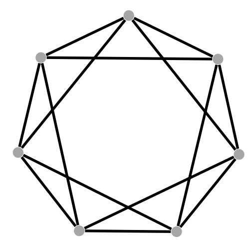
Directions: Solve 5 of the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. [CM 10.1.12] Given positive integers p and q, let $f(p, q)$ be the largest m such that there is some set of m distinct integers in the interval $[-p, q]$ not containing three integers that sum to 0. Show that $1 + \max\{p, q\} \leq f(p, q) \leq 2 + \max\{p, q\}$. (Note: it is possible to show that $f(p, q) = 1 + \max\{p, q\}$ except when p and q are the same even integer, in which case $f(p, q) = 2 + \max\{p, q\}$.)

2. [CM 10.1.30] Place n points on a circle. Let $G_{n,k}$ be the $2k$-regular graph defined on these points by joining each point to the k nearest points in each direction on the circle. For example, $G_{n,1}$ is the n-vertex cycle C_n, and $G_{7,2}$ appears below. For $k \geq 2$, prove that $\chi(G_{n,k}) > k + 2$ if $n = k(k + 1) - 1$ and $\chi(G_{n,k}) \leq k + 2$ if $n \geq k(k + 1)$.



3. [CM 10.2.9] Let S be a set of $R(m, m; 3)$ points in the plane no three of which are collinear. Prove that S contains m points that form a convex m-gon.

4. [CM 10.2.25] The graph mK_2 is the 1-regular graph on $2m$ vertices. (The notation mK_2 is used since this graph is formed from m disjoint copies of K_2.) Prove that $R(mK_2, mK_2) = 3m - 1$.

5. [CM 10.2.31] Let G be an n-vertex graph with m edges. For each vertex v in G, let $d(v)$ be the degree of v.

   (a) Prove that if $\sum_{v \in V(G)} \binom{d(v)}{2} > \binom{n}{2}$, then G contains a 4-cycle.

   (b) Prove that if $m > \frac{n}{4}(1 + \sqrt{4n - 3})$, then G contains a 4-cycle. (Note: in the language of Turán theory, this shows that $\mathrm{ex}(n; C_4) \leq \frac{n}{4}(1 + \sqrt{4n - 3}) \leq (2n^{3/2} + n)/4$.) (Hint: let $f(x) = x(x - 1)/2$. Since $f''(x) > 0$, a convexity argument shows that subject to $x_1 + \cdots + x_k = t$, the sum $f(x_1) + \cdots + f(x_k)$ is minimized when $x_1 = x_2 = \cdots = x_k = t/k$. You do not need to prove the convexity result in your homework, but you should try to prove it for yourself at least for the case $g(x) = x^2$.)

   (c) Recall that $R_k(C_4)$ is the least integer n such that every k-edge-coloring of K_n contains a monochromatic copy of C_4. Prove that $R_k(C_4) \leq k^2 + k + 2$. (Comment: using difference sets, there is a lower bound of $k^2 - k + 2$.)

6. [CM 11.2.{4,5,12}] In each of the following, prove your answer is correct.

   (a) Two permutations (in word form) **intersect** if they agree in some position. Determine the maximum size of an intersecting family of permutations of $[n]$.

   (b) Does every maximal intersecting family of subsets in $[n]$ have size 2^{n-1}?

(c) Let D_n be the digraph on n vertices that contains an edge from u to v for each ordered pair (u, v). An r-edge-coloring is **good** if, for each u, v, w with $u \neq v$ and $v \neq w$ (we allow $u = w$), the color on uv is different from the color on vw. For each r, determine the maximum n such that D_n has a good r-edge-coloring. Hint: use Sperner's Theorem (Theorem 11.2.11).