Directions: Solve 5 of the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. [CM 2.2.4] Let $a_n = 3a_{n-1} - 2a_{n-2} + 1$ for $n \geq 2$ with $(a_0, a_1) = (2, 4)$. Find a simple formula for a_n first using the characteristic equation method, and again using generating functions.

2. [CM 2.2.13] Let a_n be the number of n-tuples in $[4]^n$ that have at least one 1 and have no 2 appearing before the first 1 (note that $\langle a \rangle$ begins $0, 1, 6, \ldots$). Obtain and solve a recurrence for $\langle a \rangle$. Give a direct counting argument (without using summations) to prove the resulting simple formula.

3. [CM 2.2.31] A row of n lightbulbs must be turned on; initially, they are all off. Bulb 1 can be turned on or off at any time. For $i > 1$, bulb i can be turned on or off only when bulb $i - 1$ is on and all earlier bulbs are off. Let a_n be the number of steps needed to turn on all the lights; note that $\langle a \rangle$ begins $(0, 1, 2, 5, 10, 21, \ldots)$. Let b_n be the number of steps needed to turn on bulb n for the first time.

   (a) Find a recurrence for $\langle b \rangle$ and solve it.

   (b) Use $\langle b \rangle$ to find a recurrence for $\langle a \rangle$.

   (c) Solve the recurrence for $\langle a \rangle$.

4. [CM 2.2.33] Let a_n be the number of words of length n of the alphabet $\{0, 1, 2\}$ such that 1 and 2 are never adjacent.

   (a) Obtain a second-order recurrence relation for $\langle a \rangle$. (*Hint*: An easy approach is first find a recurrence having no fixed order. There is also a direct (more difficult) combinatorial argument.)

   (b) Solve for a_n.

5. [CM 2.3.13] In class, we argued that $n! \sim k\sqrt{n}(n/e)^n$ for some constant k. Here, we show that $k = \sqrt{2\pi}$, completing the derivation of Stirling's formula.

   (a) Let $I_m = \int_0^{\pi/2} (\sin x)^m \, dx$ for $m \in \mathbb{N}_0$. Compute I_m. *Hint:* Integrate by parts for $m \geq 2$. Set $u = (\sin x)^{m-1}$ and $dv = (\sin x)\, dx$. Try small m first if needed.

   (b) Show directly from the definition of I_m that $I_{2n-1} \geq I_{2n} \geq I_{2n+1}$. Use this to prove the **Wallis product**:
   $$\frac{\pi}{2} = \lim_{n \to \infty} \frac{2^{4n}(n!)^4}{[(2n)!]^2(2n+1)}$$

   (c) Use the Wallis product to compute $k = \sqrt{2\pi}$.

6. [CM 3.1.{3,15}] Generating Functions.

   (a) A child wants to buy candy. Four types of candy have prices two cents, one cent, two cents, and five cents per piece, respectively. Build the enumerator by total cost for the number of ways to spend money. Express the enumerator in the form $1/f(x)$, where $f(x)$ is a finite product of polynomials.

   (b) Let a_n be the number of ways to pick a nonnegative integer r, roll one six-sided die r times, and obtain a list of outcomes with sum n. (Caution: r is not fixed; for example, $(a_0, a_1, a_2, a_3) = (1, 1, 2, 4)$.) Express the generating function for $\langle a \rangle$ as the ratio of two polynomials that each have at most three terms. Obtain a recurrence for $\langle a \rangle$ directly from the generating function. Give a combinatorial argument for the recurrence.