Math571 **Homework 1** **Due January 28, 2012**

Directions: Solve 5 of the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines that apply to all homeworks.

1. [CM 1.1.{7,15}] Recall that $[n] = \{1, 2, \ldots, n\}$.

   (a) Count the subsets of $[n]$ that contain at least one odd integer.

   (b) Count the k-sets in $[n]$ having no two consecutive integers.

   (c) Count the lists of subsets $A_0, A_1, \ldots, A_n$ of $[n]$ such that $A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n$. Count the lists such that $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n$.

   (d) Use binomial coefficients to prove that $n^3 + 5n$ is divisible by 6 for every integer n. *Hint*: express $n^3 + 5n$ as the sum of binomial coefficients.

2. [CM 1.1.18] Count the lists of m ones and n zeros that have exactly k runs of ones, where a **run** is a maximal set of consecutive entries with the same value.

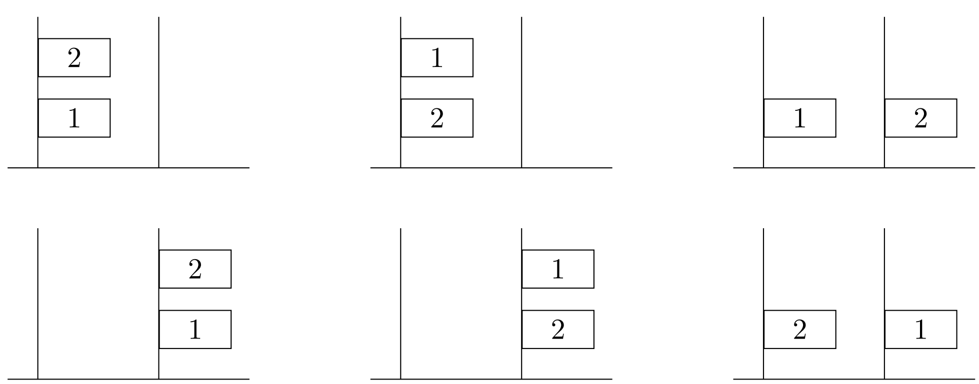
3. [CM 1.1.19] Recall that $[3]^n = \{1, 2, 3\}^n = \{(a_1, \ldots, a_n)\colon 1 \le a_i \le 3 \text{ for each } i\}$. Give a summation for the number of elements of $[3]^n$ with k odd entries that have no consecutive 1 and 3.

4. [CM 1.1.28] For a permutation π of $[n]$, the **displacement** of π is $\sum_{i=1}^{n} |\pi(i) - i|$. Prove that the largest displacement of a permutation of $[n]$ is $\lfloor n^2/2 \rfloor$.

5. [CM 1.1.31] A permutation is **graceful** if the absolute differences between successive elements are distinct. Prove that if the set of elements in even-indexed positions of a graceful permutation of $[2n]$ is $[n]$, then the first and last elements differ by n. *Hint*: Let π be a graceful permutation of $[2n]$ such that $\pi(i) \le n$ if and only if i is even. Evaluate $\sum_{i=1}^{n} |\pi(i) - \pi(i+1)|$ in two different ways, where all addition is modulo n (including $\pi(n+1) = \pi(1)$).

6. [CM 1.1.34] Flags on poles.

   (a) Obtain a simple formula for the number of ways to put m distinct flags on a row of r flagpoles. Poles may be empty, and changing the order of flags on a pole changes the arrangement. The formula must only use one "m" and one "r". (The answer is 6 for $m = r = 2$, as shown below.)



   (b) Prove that the identity below for rising factorials hods for all $x, y \in \mathbb{R}$.

$$(x+y)^{(n)} = \sum_{k} \binom{n}{k} x^{(k)} y^{(n-k)}$$