Name: Solutions

Directions: Show all work. No credit for answers without work. The test has 5 pages, with 10 points per page. **Your lowest scoring page will be dropped.** *Recommendation:* to save on time, decide now which page you will drop and work on the other 4 pages.

1. [**10 points**] A population of animals has a birth rate β and death rate δ that are proportional to P. Recall that this means that $\beta = \lambda_1 P$ and $\delta = \lambda_2 P$ for some constants λ_1 and λ_2. Also, recall that β represents births per unit population per unit time, and δ represents deaths per unit population per unit time.

   (a) Show that $P(t) = \dfrac{P_0}{1 - kP_0 t}$ where $k = \lambda_1 - \lambda_2$.

$$\Delta P \approx \beta \cdot P \cdot \Delta t - \delta \cdot P \cdot \Delta t$$

$$\frac{dP}{dt} = (\beta - \delta)P = (\lambda_1 P - \lambda_2 P)P = kP^2$$

$$\int \frac{1}{P^2}\, dP = \int k\, dt$$

$$-P^{-1} = kt + C$$

$$-\frac{1}{P_0} = \cancel{k}0 + C$$

$$-\frac{1}{P} = kt - \frac{1}{P_0}$$

$$-\frac{1}{kt - \frac{1}{P_0}} = P$$

$$P = \frac{1}{\frac{1}{P_0} - kt}$$

$$P = \frac{P_0}{1 - kP_0 t}$$

   (b) When $\lambda_1 > \lambda_2$, the population experiences "doomsday". When does doomsday occur? Find a general formula.

Doomsday: when $P \to \infty$. When denominator $\to 0^+$.

$$0 = 1 - kP_0 t$$

$$kP_0 t = 1$$

$$t = \boxed{\frac{1}{kP_0}}$$

2. [**10 points**] Consider the differential equation $\frac{dP}{dt} = \frac{1}{2}P(11-P)$ with $P(0) = 3$.

(a) Using the fact that $P(t)$ is a logistic differential equation, state the solution to $P(t)$. It is not necessary to show any work. *Note:* you may solve this equation directly and copy your answer here, but this will cost you time.

$$P = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}} = \frac{11\cdot 3}{3 + (11-3)e^{-\frac{1}{2}\cdot 11\cdot t}} = \boxed{\frac{33}{3+8e^{-11/2\, t}}}$$

(b) Use Euler's method with step size $h = 1/3$ to approximate $P(t)$ over the interval $[0, 1]$.

- $P(0) = \boxed{3} = Y_0$

$\Rightarrow$ Slope $= \frac{1}{2}\cdot 3\cdot(11-3) = \frac{1}{2}\cdot 3\cdot 8 = 12$

- $P(\frac{1}{3}) \approx \overset{P(0)}{\cancel{Y_0}} + h\cdot 12 = 3 + \frac{1}{3}\cdot 12 = 3 + 4 = \boxed{7}$

$\Rightarrow$ slope $= \frac{1}{2}\cdot 7\cdot(11-7) = \frac{1}{2}\cdot 7\cdot 4 = 14$

- $P(\frac{2}{3}) \approx P(\frac{1}{3}) + h\cdot 14 = 7 + \frac{1}{3}\cdot 14 \approx \boxed{11.667}$

$\Rightarrow$ slope $= \frac{1}{2}\cdot 11.667\,(11 - 11.667) \approx -\frac{35}{9} \approx -3.89$

- $P(1) \approx P(2/3) + h\cdot(-3.89) \approx \boxed{10.37}$

(c) Other than the error, what is particularly unfortunate about the estimate for $P(2/3)$?

P(2/3) is estimated to exceed 11, which is the limiting population/carrying capacity.

3. [**5 points**] Find the Wronskian of $f_1(x) = 5x$, $f_2(x) = \sin(x)$, and $f_3(x) = e^x$. Are these functions linearly dependent or linearly independent? Explain how you know.

$$W = \begin{vmatrix} 5x & \sin x & e^x \\ 5 & \cos x & e^x \\ 0 & -\sin x & e^x \end{vmatrix} = 5x \begin{vmatrix} \cos x & e^x \\ -\sin x & e^x \end{vmatrix} - 5 \begin{vmatrix} \sin x & e^x \\ -\sin x & e^x \end{vmatrix} + 0$$

$$= 5x\left(e^x \cos x + e^x \sin x\right) - 5\left(\sin x\, e^x + e^x \sin x\right)$$

$$= 5xe^x(\cos x + \sin x) - 10e^x \sin x$$

$$= 5e^x(x\cos x + x\sin x - 2\sin x) = \boxed{5e^x(x\cos x + (x-2)\sin x)}$$

Since $W \neq 0$, these functions are linearly ~~dependent~~ independent.

4. [**5 points**] Find general solutions to the following differential equation: $y^{(3)} + 8y'' + 16y' = 0$.

$$r^3 + 8r^2 + 16r = 0$$

$$r(r^2 + 8r + 16) = 0$$

$$r(r+4)(r+4) = 0$$

$$r = 0,\ r = -4,\ r = -4$$

$$\boxed{y = c_1 + c_2 e^{-4x} + c_3 x e^{-4x}}$$

5. [**10 points**] Solve the following initial value problem: $y'' - 6y' + 8y = 0$ with $y(0) = 2$ and $y'(0) = 5$.

$$r^2 - 6r + 8 = 0$$

$$(r-4)(r-2) = 0$$

$$y = c_1 e^{4x} + c_2 e^{2x} \qquad y' = 4c_1 e^{4x} + 2c_2 e^{2x}$$

$\underline{y(0)}$: $2 = c_1 + c_2$

$\underline{y'(0)}$: $5 = 4c_1 + 2c_2$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 4 & 2 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & \frac{1}{2} & 5/4 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -\frac{1}{2} & -3/4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 3/2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 3/2 \end{array}\right] \qquad c_1 = \frac{1}{2},\ c_2 = 3/2$$

$$\boxed{y = \frac{1}{2} e^{4x} + \frac{3}{2} e^{2x}}$$

6. **[10 points]** A 5 kg-mass is attached to a spring with sprin~~t~~g constant $k = 290$ N/m. Initially, the spring is stretched 3 m from its equilibrium position and is traveling away from equilibrium at 19 m/s. The motion of the spring is damped with damping constant $c = 30$ Ns/m.

(a) Express the position function of the mass in the form $x(t) = e^{-kt}(A\cos(\omega t) + B\sin(\omega t))$.

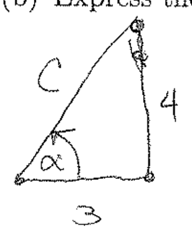
$$mx'' + cx' + kx = 0$$
$$5x'' + 30x' + 290x = 0$$
$$x'' + 6x' + 58x = 0$$
$$r^2 + 6r + 58 = 0$$
$$r = \frac{-6 \pm \sqrt{36 - 4 \cdot 58}}{2}$$
$$= -3 \pm 7i$$

$$x = e^{(-3+7i)t}$$
$$= e^{-3t} \cdot e^{7ti}$$
$$= e^{-3t}(\cos(7t) + i\sin(7t))$$
$$x = e^{-3t}(A\cos(7t) + B\sin(7t))$$
$$x' = -3e^{-3t}(A\cos(7t) + B\sin(7t)) + e^{-3t}(-7A\sin(7t) + 7B\cos(7t))$$
$x(0) = 3$: $3 = A$

$x'(0) = 19$: $19 = -3(A) + (7B)$; $B = 4$.

$$\boxed{x = e^{-3t}(3\cos(7t) + 4\sin(7t))}$$

(b) Express the position function of the mass in the form $x(t) = e^{-kt}(C\cos(\omega t - \alpha))$.



$$C = \sqrt{3^2 + 4^2} = 5$$
$$\alpha = \arctan\frac{4}{3} = 0.927$$

$$\boxed{x = e^{-3t}(5\cos(7t - 0.927))}$$

(c) What is the frequency of the motion of the mass? Include units.

$\omega = 7$ rad/sec, so $\nu = \frac{7}{2\pi}$ Hz $= \boxed{1.114 \text{ Hz}}$