Name: Solutions

Directions: Show all work. No credit for answers without work. Primes denote derivatives with respect to x.

1. [**5 points**] Give a differential equation that models the following situation. In a town with 4 million people, the rate at which ~~number of~~ people ~~that~~ hear a rumor is proportional to the number of people who have not heard the rumor. Use $N(t)$ for the number of people (in millions) who have heard the rumor at time t.

$$\frac{dN}{dt} = k(4-N),$$ where k is a positive const.

2. [**5 points**] Verify that $y = \frac{1}{1+x^2}$ is a solution to the differential equation $y' + 2xy^2 = 0$.

$$y = (1+x^2)^{-1}$$

$$y' = -(1+x^2)^{-2} \cdot 2x$$

$$= -\frac{2x}{(1+x^2)^2}$$

$$y' + 2xy^2 = -\frac{2x}{(1+x^2)^2} + 2xy^2$$

$$= -\frac{2x}{(1+x^2)^2} + \frac{2x}{(1+x^2)^2}$$

$$= 0 \checkmark$$

3. [**10 points**] On another planet, a ball dropped from a height of 5 meters takes 3 seconds to hit the ground. Find the amount of time it takes for the ball to hit the ground if it is dropped from a height of 120 meters.

- Let g be the acceleration due to gravity on the other planet.

- $\frac{dv}{dt} = -g$
- $v = -gt + v_0$
- $\frac{dx}{dt} = -gt + v_0$
- $x = \frac{-g}{2}t^2 + v_0 t + x_0$

$\Rightarrow$ When dropped from 5 meters:

$v_0 = 0$, $x_0 = 5$ and $x(3) = 0$.

$$x = \frac{-g}{2}t^2 + \cancel{0} \cdot t + 5$$

$$0 = \frac{-g}{2}(3)^2 + 5$$

$$g = \frac{10}{9} \ m/s^2$$

$\Rightarrow$ When dropped from 120 meters:

- Let t_* be time of impact.
- $v_0 = 0$, $x_0 = 120$, $x(t_*) = 0$

$$x = \frac{-g}{2}t^2 + \cancel{0} \cdot t + 120$$

$$0 = -\frac{10}{18}t_*^2 + 120$$

$$t_*^2 = 12 \cdot 18 = 216$$

$$t_* \approx \boxed{14.7 \text{ seconds}}$$

4. [**5 points**] Solve the initial value problem $y^2y' = e^x(y^3+1)$ and $y(0)=1$.

Seperable:

$$\frac{y^2}{y^3+1}\, y' = e^x$$

$$\int \frac{y^2}{y^3+1}\,dy = \int e^x\,dx$$

$$\frac{1}{3}\int \frac{1}{y^3+1}\cdot 3y^2\,dy = e^x + C$$

$$\frac{1}{3}\ln|y^3+1| = e^x + C$$

$y(0)=1$: $\frac{1}{3}\ln(2) = e^0 + C$

$$C = \frac{\ln(2)}{3} - 1$$

$$\frac{1}{3}\ln(y^3+1) = e^x + C$$

$$\ln(y^3+1) = 3(e^x+C)$$

$$y^3+1 = e^{3(e^x+C)}$$

$$\boxed{y = \left(e^{3(e^x+C)} - 1\right)^{1/3}}$$

$$y = \left(e^{3C}\cdot e^{3e^x} - 1\right)^{1/3}$$

$$y = \left(e^{\ln(2)-3}\cdot e^{3e^x} - 1\right)^{1/3}$$

$$\boxed{y = \left(\frac{2}{e^3}\cdot e^{3e^x} - 1\right)^{1/3}}$$

5. [**5 points**] Find the general solution to $xy + y^2 - x^2y' = 0$.

Homogeneous:

$$x^2y' = xy + y^2$$

$$y' = \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

Use $v = \frac{y}{x}$; $y = vx$; $y' = v'x + v$.

$$v'x + \cancel{v} = \cancel{v} + v^2$$

$$\frac{1}{v^2}\cdot v' = \frac{1}{x}$$

$$\int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx$$

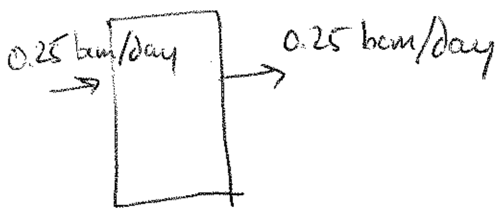
$$-v^{-1} = \ln|x| + C$$

$$-\frac{1}{\left(\frac{y}{x}\right)} = \ln|x| + C$$

$$-\frac{x}{\ln|x|+C} = y$$

$$\boxed{y = \frac{-x}{\ln|x|+C}}$$

6. [**10 points**] A city's water reservoir contains 8 billion cubic meters (bcm) of water. The purification system ensures that the concentration of pollutants remains constant at 0.5 kilograms per bcm, and sensors will trigger an alarm if the concentration of pollutants rises above 1 kilogram per bcm. Water flows in and out of the reservoir at the same rate of 0.25 bcm per day, and the concentration of pollutants in the inflow is 2 kilograms per bcm. At all times, the reservoir is well mixed. If the purification system fails, how much time elapses before the alarm is triggered?



- Let $t=0$ be the time when purification fails.
- Let $x(t)$ be the amount of pollutants in the reservoir at time t. Volume is always 8 bcm.

- $\Delta x = (0.25)\cdot(2)\cdot\Delta t - \frac{x}{V}\cdot(0.25)\cdot\Delta t$

- $\frac{dx}{dt} = \frac{1}{2} - \frac{x}{8}\cdot\frac{1}{4}$

- $\frac{dx}{dt} + \frac{1}{32}x = \frac{1}{2}$, $\rho = e^{\int \frac{1}{32}\, dt} = e^{\frac{1}{32}t}$

- $e^{\frac{1}{32}t}\frac{dx}{dt} + \frac{1}{32}xe^{\frac{1}{32}t} = \frac{1}{2}e^{\frac{1}{32}t}$

$$\frac{d}{dt}\left[e^{\frac{1}{32}t}x\right] = \frac{1}{2}e^{\frac{1}{32}t}$$

$$e^{\frac{1}{32}t}x = \int \frac{1}{2}e^{\frac{1}{32}t}\, dt$$

$$e^{\frac{1}{32}t}x = \frac{32}{2}\cdot e^{\frac{1}{32}t} + C$$

$$x = 16 + Ce^{-\frac{1}{32}t}$$

- Initially, there are 4 kg of pollutant $(0.5\cdot 8 = 4)$:

$$4 = 16 + Ce^{-\frac{1}{32}\cdot 0}$$

$$4 = 16 + C, \quad C = -12$$

- $x = 16 - 12e^{-\frac{1}{32}t}$

- Alarm when $x = 8$:

$$8 = 16 - 12e^{-\frac{1}{32}t}$$

$$12e^{-\frac{1}{32}t} = 8$$

$$-\frac{1}{32}t = \ln\frac{2}{3}$$

$$t = -32\ln\frac{2}{3} \approx \boxed{12.97 \text{ days}}$$

7. [**5 points**] Verify that $\overbrace{(2xy^3+e^x)}^{M}dx+\overbrace{(3x^2y^2+\sin y)}^{N}dy=0$ is an exact differential equation and solve.

$M_y = 2x\cdot 3y^2 = 6xy^2$
$N_x = 6xy^2$ ✓. Exact.

$F = \int M\,dx + g(y)$

$= \int 2xy^3+e^x\,dx + g(y)$

$= x^2y^3+e^x+g(y)$

Set $F_y = N$:

$\cancel{x^2\cdot 3y^2} + 0 + g'(y) = \cancel{3x^2y^2} + \sin y$

$g'(y) = \sin y$

$g = \int \sin y\,dy$

$= -\cos y + C$

$\boxed{F(x,y) = x^2y^3+e^x-\cos y + C. \quad x^2y^3+e^x-\cos y = C}$

8. [**5 points**] Find a two-parameter family solution to $xy''+y'=4x$.

Let $p(x)=y'$, $\frac{dp}{dx}=y''$:

$x\frac{dp}{dx}+p=4x$

$\frac{dp}{dx}+\frac{1}{x}p=4 \qquad \rho = e^{\int\frac{1}{x}dx}=x$

$x\frac{dp}{dx}+p=4x$

$\frac{d}{dx}[xp]=4x$

$xp=\int 4x\,dx$

$xp=2x^2+A$

$p=2x+\frac{A}{x}$

$y'=2x+\frac{A}{x}$

$y=\int 2x+\frac{A}{x}\,dx$

$\boxed{y = x^2+A\ln|x|+B}$