Name: Solutions

Directions: Show all work. No credit for answers without work.

1. [**3 points**] Solve the following initial value problem: $\frac{dx}{dt} = 7x(x-13)$, and $x(0) = 17$.

$$\int \frac{1}{7x(x-13)}\,dx = \int dt$$

$$\frac{A}{7x} + \frac{B}{x-13} = \frac{1}{7x(x-13)}$$

$$A(x-13) + B\cdot 7x = 1$$

$\underline{x=13}$: $B\cdot 7\cdot 13 = 1$, $B = \frac{1}{7\cdot 13}$

$\underline{x=0}$: $A\cdot(-13) = 1$, $A = -\frac{1}{13}$

$$\int -\frac{1}{13\cdot 7}\cdot\frac{1}{x}\,dx + \int \frac{1}{13\cdot 7}\cdot\frac{1}{(x-13)}\,dx = t + C$$

$$-\frac{1}{13\cdot 7}\ln|x| + \frac{1}{13\cdot 7}\ln|x-13| = t + C$$

$$\frac{1}{13\cdot 7}\ln\left|\frac{x-13}{x}\right| = t + C$$

$\underline{x(0)=17}$: $C = \frac{1}{13\cdot 7}\ln\left(\frac{4}{17}\right)$

$$\ln\frac{x-13}{x} = 13\cdot 7(t) + \ln\left(\frac{4}{17}\right)$$

$$\frac{x-13}{x} = \frac{4}{17}\cdot e^{13\cdot 7t} \quad \boxed{x = \frac{13}{1-\frac{4}{17}e^{13\cdot 7t}}}$$

2. [**2 points**] Recall the logistic differential equation $\frac{dx}{dt} = kx(M-x)$. A population of cows satisfying the logistic equation initially has 4000 members and is growing at a rate of 3 cows per day. The environment can support a population of 10,000 cows.

(a) Solve for k in the logistic differential equation.

$x_0 = 4000$

$M = 10{,}000$

$\underline{t=0}$: $\frac{dx}{dt} = 3$

$\underline{t=0}$: $3 = k\cdot 4000\,(10{,}000 - 4000)$

$$\boxed{k = \frac{3}{24{,}000{,}000}}$$

(b) Explicitly give the formula for $x(t)$.

$$x = \frac{Mx_0}{x_0 + (M-x_0)e^{-tMt}} = \frac{40{,}000{,}000}{4000 + 6000e^{-\frac{3}{24{,}000{,}000}\cdot 10{,}000t}}$$

$$= \frac{40{,}000{,}000}{4000 + 6000e^{-\frac{3}{2400}t}} = \boxed{\frac{40{,}000{,}000}{4000 + 6000e^{-t/800}}}$$

3. [**3 points**] Find the equilibrium solutions to $\dfrac{dx}{dt} = x(x^2-4)$. Use a phase diagram to classify each equilibrium solution as stable, semi-stable, or unstable.

$x(x^2-4)=0$

$x(x+2)(x-2)=0$

$x=0,\ x=-2,\ x=2.$

Phase diagram:

$x=2$: unstable

$x=0$: stable

$x=-2$: unstable

4. [**2 points**] Give the bifurcation diagram for the differential equation $\dfrac{dx}{dt} = x + kx^3$; this is the plot of all points (k, c) such that $x = c$ is an equilibrium solution to $\dfrac{dx}{dt} = x + kx^3$.

$0 = x + kx^3$

$= x(1+kx^2)$

$x=0$ or $0 = 1 + kx^2$

$x^2 = -\dfrac{1}{k}$

$x = \pm\sqrt{-\dfrac{1}{k}}$