

Name: Solutions

Directions: Show all work. No credit for answers without work.

1. [3 points] Solve the following initial value problem: $\frac{dx}{dt} = 7x(x-13)$, and $x(0) = 17$.

$$\int \frac{1}{7x(x-13)} dx = \int dt$$

$$\frac{A}{7x} + \frac{B}{x-13} = \frac{1}{7x(x-13)}$$

$$A(x-13) + B \cdot 7x = 1$$

$$x=13: B \cdot 7 \cdot 13 = 1, B = \frac{1}{7 \cdot 13}$$

$$x=0: A \cdot (-13) = 1, A = -\frac{1}{13}$$

$$\int -\frac{1}{13 \cdot 7} \cdot \frac{1}{x} dx + \int \frac{1}{13 \cdot 7} \cdot \frac{1}{(x-13)} dx = t + C$$

$$-\frac{1}{13 \cdot 7} \ln|x| + \frac{1}{13 \cdot 7} \ln|x-13| = t + C$$

$$\frac{1}{13 \cdot 7} \ln \left| \frac{x-13}{x} \right| = t + C$$

$$x(0)=17: C = \frac{1}{13 \cdot 7} \ln \left(\frac{4}{17} \right)$$

$$\ln \frac{x-13}{x} = 13 \cdot 7(t) + \ln \left(\frac{4}{17} \right)$$

$$\frac{x-13}{x} = \frac{4}{17} \cdot e^{13 \cdot 7t} \quad \boxed{x = \frac{13}{1 - \frac{4}{17} e^{13 \cdot 7t}}}$$

2. [2 points] Recall the logistic differential equation $\frac{dx}{dt} = kx(M-x)$. A population of cows satisfying the logistic equation initially has 4000 members and is growing at a rate of 3 cows per day. The environment can support a population of 10,000 cows.

- (a) Solve for k in the logistic differential equation.

$$t=0: 3 = k \cdot 4000 (10,000 - 4000)$$

$$\boxed{k = \frac{3}{24,000,000}}$$

$$x_0 = 4000$$

$$M = 10,000$$

$$t=0: \frac{dx}{dt} = 3$$

- (b) Explicitly give the formula for $x(t)$.

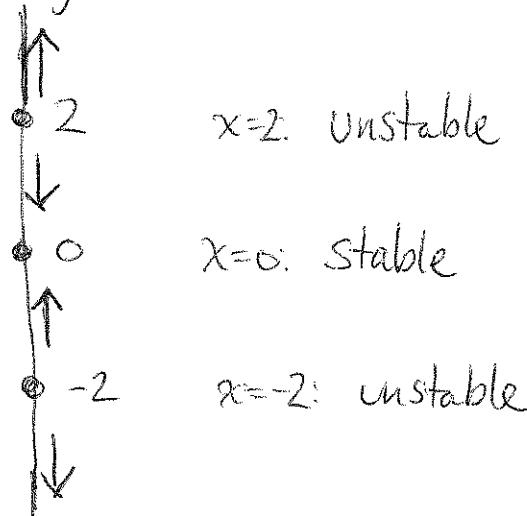
$$x = \frac{Mx_0}{x_0 + (M-x_0)e^{-kt}} = \frac{40,000,000}{4000 + 6000e^{-\frac{3}{24,000,000} \cdot 10,000t}}$$

$$= \frac{40,000,000}{4000 + 6000e^{-\frac{3}{2400}t}} = \boxed{\frac{40,000,000}{4000 + 6000e^{-\frac{t}{800}}}}$$

3. [3 points] Find the equilibrium solutions to $\frac{dx}{dt} = x(x^2 - 4)$. Use a phase diagram to classify each equilibrium solution as stable, semi-stable, or unstable.

$$\begin{aligned}x(x^2 - 4) &= 0 \\x(x+2)(x-2) &= 0 \\x=0, x=-2, x=2.\end{aligned}$$

Phase diagram:



4. [2 points] Give the bifurcation diagram for the differential equation $\frac{dx}{dt} = x + kx^3$; this is the plot of all points (k, c) such that $x = c$ is an equilibrium solution to $\frac{dx}{dt} = x + kx^3$.

$$\begin{aligned}0 &= x + kx^3 \\&= x(1 + kx^2)\end{aligned}$$

$$x=0 \text{ or } 0=1+kx^2$$

$$x^2 = -\frac{1}{k}$$

$$x = \pm \sqrt{-\frac{1}{k}}$$

