Name: Solution

Directions: Show all work. No credit for answers without work. Primes denote derivatives with respect to x.

1. [**5 points**] Solve the following.

(a) $xy' - y = x^2$, $y(2) = 10$.

$$y' - \frac{1}{x}y = x \qquad \rho = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}.$$

$$\frac{1}{x}y' - \frac{1}{x^2}y = 1$$

$$\frac{d}{dx}\left[\frac{1}{x}y\right] = 1$$

$$\frac{1}{x}y = \int 1\,dx$$

$$\frac{1}{x}y = x + C$$

$$y = x^2 + Cx$$

$$10 = 4 + 2C, \qquad C = 3$$

$$\boxed{y = x^2 + 3x}$$

(b) $3y^2y' + y^3 = e^{-x}$

① $$y' + \frac{1}{3}y = \frac{e^{-x}}{3}\cdot\frac{1}{y^2}$$

$$y' + \frac{1}{3}y = \frac{e^{-x}}{3}\cdot y^{-2}$$

$$\frac{1}{3}v^{-2/3}v' + \frac{1}{3}v^{\frac{1}{3}} = \frac{e^{-x}}{3}\cdot v^{-2/3}$$

② Bernoulli: $v = y^{1-n}$

$$v = y^{1-(-2)}$$

$$v = y^3$$

$$y = v^{\frac{1}{3}}$$

$$y' = \frac{1}{3}v^{-2/3}v'$$

③ $$v' + v = e^{-x}, \quad \rho = e^{\int 1 dx} = e^x$$

$$e^x v' + e^x v = 1$$

$$\frac{d}{dx}\left[e^x v\right] = 1$$

$$e^x v = x + C$$

④ $$e^x y^3 = x + C$$

$$y = \left[(x+C)e^{-x}\right]^{\frac{1}{3}}$$

$$\boxed{y = (x+C)^{1/3}e^{-x/3}}$$

2. **[2.5 points]** Solve: $x^2y' = xy + x^2e^{y/x}$.

Homogeneous:

$V = \frac{y}{x}, \quad y = vx, \quad y = v'x + v$

$$y' = \frac{y}{x} + e^{y/x}$$

$$v'x + \cancel{v} = \cancel{v} + e^v$$

$$\frac{1}{e^v} v' = \frac{1}{x}$$

$$\int \frac{1}{e^v}\, dv = \int \frac{1}{x}\, dx$$

$$-e^{-v} = \ln x + C$$

$$e^{-v} = C - \ln x$$

$$-v = \ln(C - \ln(x))$$

$$v = -\ln(C - \ln(x))$$

$$\frac{y}{x} = \quad ''$$

$$\boxed{y = -x\ln(C - \ln(x))}$$

3. **[2.5 points]** Verify that the following differential equation is exact, and solve.
$(4x - y)dx + (6y - x)dy = 0.$

$\underset{M}{\uparrow} \qquad \underset{N}{\uparrow}$

Verify: $M_y = -1, \quad N_x = -1$ ✓

Solve: $F = \int M\, dx + g(y)$

$$= \int 4x - y\, dx + g(y)$$

$$= 2x^2 - yx + g(y).$$

$F_y = N$: $\cancel{-x} + g'(y) = 6y \cancel{-x}$

$$g(y) = \int 6y\, dy$$

$$g(y) = 3y^2 + C$$

$\boxed{F = 2x^2 - yx + 3y^2 + C, \text{ and } 2x^2 - yx + 3y^2 = C}$

is the implicit solution.