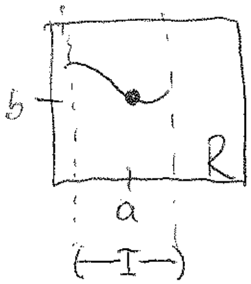
Name: Solution.

**Directions:** Show all work. No credit for answers without work.

1. [**2 points**] Consider the general first-order differential equation $\frac{dy}{dx} = f(x,y)$ with initial value $y(a) = b$. State the theorem which gives conditions under which a solution exists and is unique.




If $f$ and $f_y$ are continuous on some rectangle $R$ containing $(a,b)$, then ~~the~~ a solution exists and is unique on some open interval $I$ containing $a$.

2. [**2 parts, 1 point each**] For each of the following initial value problems, determine whether the above theorem guarantees existence and uniqueness.

(a) $\frac{dy}{dx} = \frac{x^3(y-1)}{x+1}$, and $y(2) = 1$.

- $f = \frac{x^3}{x+1} \cdot (y-1)$, continuous everywhere except $x = -1$
- $f_y = \frac{x^3}{x+1} \cdot 1$, continuous everywhere except $x = -1$.

So, the $\boxed{\text{theorem guarantees existence and uniqueness.}}$

(b) $\frac{dy}{dx} = \frac{x^3\sqrt{y-1}}{x+1}$, and $y(2) = 1$.

- $f = \frac{x^3}{x+1}\sqrt{y-1}$ continuous everywhere except $x = -1$.
- $f_y = \frac{x^3}{x+1} \cdot \frac{1}{2}(y-1)^{-\frac{1}{2}}$

$= \frac{x^3}{x+1} \cdot \frac{1}{2\sqrt{y-1}}$ continuous everywhere except $x = -1$ or $y \le 1$.

- So, at $(2,1)$, $\boxed{\text{existence and uniqueness are not guaranteed.}}$

3. [**3 points**] Solve the following initial value problem: $\frac{dy}{dx} = \frac{x^2}{e^y}$, and $y(3) = 0$.

$$e^y \frac{dy}{dx} = x^2$$

$$\int e^y \, dy = \int x^2 dx$$

$$e^y = \frac{x^3}{3} + C$$

$[y(3)=0]$:

$$e^0 = \frac{3^3}{3} + C$$

$$1 = 9 + C$$

$$C = -8$$

$$e^y = \frac{x^3}{3} - 8$$

$$\boxed{y = \ln\left(\frac{x^3}{3} - 8\right)}$$

4. [**3 points**] Find the general solution to $\frac{dy}{dx} = y\sin(x) + \sin(x)$.

$$\frac{dy}{dx} = (y+1)\sin(x)$$

$$\frac{1}{y+1}\frac{dy}{dx} = \sin(x)$$

$$\int \frac{1}{y+1} dy = \int \sin(x)\, dx$$

$$\ln(y+1) = -\cos(x) + C$$

$$y + 1 = Ce^{-\cos(x)}$$

$$\boxed{y = Ce^{-\cos(x)} - 1}$$