Math 242 Example from Jan 25, 2012

Ex: Find a general solution to $(x^2+1)\dfrac{dy}{dx} + 3xy = 6x+1$.

Soln: This is a linear first-order diff eq. Divide by (x^2+1):

$$\frac{dy}{dx} + \frac{3x}{x^2+1}\,y = \frac{6x+1}{x^2+1}$$

- Compute integrating factor:

$$\rho = e^{\int \frac{3x}{x^2+1}\,dx} = e^{\frac{3}{2}\int \frac{1}{x^2+1}\,2x\,dx} = e^{\frac{3}{2}\ln(x^2+1)}$$

$$= (x^2+1)^{3/2}.$$

- Multiply both sides by ρ:

$$(x^2+1)^{3/2}\frac{dy}{dx} + 3x(x^2+1)^{\frac{1}{2}}y = (6x+1)(x^2+1)^{1/2}$$

$$\frac{d}{dx}\left[(x^2+1)^{3/2}\,y\right] = (6x+1)(x^2+1)^{1/2}$$

$$(x^2+1)^{3/2}\,y = \int (6x+1)\sqrt{x^2+1}\,dx$$

$$(x^2+1)^{3/2}y = \int 6x\sqrt{x^2+1}\,dx + \int \sqrt{1+x^2}\,dx$$

$$= 3\int \sqrt{x^2+1}\cdot 2x\,dx + \int\sqrt{1+x^2}\,dx$$

$$= 3\cdot\frac{2}{3}(x^2+1)^{3/2} + \int\sqrt{1+x^2}\,dx$$

$$= 2(x^2+1)^{3/2} + \int\sqrt{1+x^2}\,dx.$$

⇒ So, we need to solve $\int\sqrt{1+x^2}\,dx$. This integral is tricky. The solution is:

$$\int\sqrt{1+x^2}\,dx = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left|x+\sqrt{1+x^2}\right| + C.$$

You can find this in ~~all~~ the integral table in the back of the book, on the ~~cover~~ inside cover.

⇒ Therefore:

$$(x^2+1)^{3/2}y = 2(x^2+1)^{3/2} + \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left|x+\sqrt{1+x^2}\right| + C$$

$$\boxed{y = 2 + \frac{x}{2(x^2+1)} + \frac{\ln\left|x+\sqrt{1+x^2}\right|}{2(x^2+1)^{3/2}} + \frac{C}{(x^2+1)^{3/2}}}$$

Note: As $x \to \infty$, $y \to 2$, regardless of C.

So, all solutions (regardless of the initial condition)

approach the solution

$$y = 2 + \frac{x}{2(x^2+1)} + \frac{\ln\left|x + \sqrt{1+x^2}\right|}{2(x^2+1)^{3/2}},$$

where $C = 0$.

~~This solution is called~~

Note: For the original equation: $(x^2+1)\frac{dy}{dx} + 3xy = 6x$,

the general solution is $y = 2 + \frac{C}{(x^2+1)^{3/2}}$,

and all solutions approach the constant solution

$y = 2$. ~~This is~~ A constant solution which

all others approach is called ~~an~~ ~~equilibrium~~ stable

equilibrium solution. (A constant solution to a diff eqn is called an equilibrium solution.)

- Final notes about $\int \sqrt{1+x^2}\, dx$:

  - This integral is too difficult to appear on a quiz or test.

  - If it ever arises in homework, it would be OK to look it up in an integral table in the back of the book.

  - You should know at least all the elementary forms from the book's table, as well as the first few trig forms:

    #1 - #22.

  - ~~The derivation~~ A solution for this integral does use the trig substitution $u = \tan x$, which transforms the integral $\int \sqrt{1+x^2}\, dx = \int \sec^3(u)\, du$.

  - Next, $\int \sec^3(u)\, du$ is solved with a "reduction formula", which is solved via integration by parts. (For more info, see me.)