Math343 Test 3 Nov 18, 2022 ~~Nov 18, 2020~~

Name: Solutions

Directions: Show all work. No credit for answers without work.

25

1. [~~10~~ 15 **points**] Use cofactor expansion to compute the determinant of the following matrix as efficiently as possible.

$$A = \begin{bmatrix} 0 & 0 & 3 & 0 & 1 \\ 0 & 2 & 5 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 7 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = (-7)\begin{vmatrix} 0 & 3 & 0 & 1 \\ 2 & 5 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (-7)(1)\begin{vmatrix} 0 & 3 & 0 \\ 2 & 5 & 0 \\ 1 & 0 & 2 \end{vmatrix} = (-7)(1)(-3)\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= 21(4-0) = 21 \cdot 4 = \boxed{84}$$

2. [**10 points**] Let A be an invertible $n \times n$-matrix.

5 (a) What can you say about the rank of A?

By IVT, $\boxed{\text{rank}(A) = n.}$

5 (b) Suppose that C is an $n \times n$ matrix obtained from A by changing one row of A. What can you say about the rank of C? Why?

The rank of A is the dimension of its row and column space. Since A is invertible, its columns are lin dep. After changing one column, the remaining cols are still linearly indep, so C has rank $\boxed{\text{at least } n-1.}$ It may still have rank n. So $\boxed{\text{rank}(C) \in \{n-1, n\}}$.

25 ~~20~~

3. [~~20~~ 25 points] Using standard techniques from class, diagonalize the following matrix A. That is, if possible, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. The matrix P^{-1} need not be computed explicitly. If diagonalization is not possible, then explain why.

$$\begin{bmatrix} 2 & 3 & -3 \\ -3 & -4 & 3 \\ -3 & -3 & 2 \end{bmatrix}$$

$$0 = \begin{vmatrix} 2-\lambda & 3 & -3 \\ -3 & -4-\lambda & 3 \\ -3 & -3 & 2-\lambda \end{vmatrix} = \left((2-\lambda)^2(-4-\lambda) + -3^3 \quad -3^3\right) - \left(3^2(-4-\lambda) + -3^2(2-\lambda) \quad -3^2(2-\lambda)\right)$$

$$= \left(-(4-4\lambda+\lambda^2)(4+\lambda) - 2\cdot 3^3\right) + 3^2(4+\lambda) + 2\cdot 3^2(2-\lambda)$$

$$= -(4+\lambda)\left[\lambda^2 - 4\lambda + 4 - 3^2\right] - 2\cdot 3^3 + 4\cdot 3^2 - 18\lambda$$

$$= ~~-[\lambda^3 - 12\lambda~~ = -(4+\lambda)\left[\lambda^2 - 4\lambda - 5\right] - 18\lambda + 3^2(4-6)$$

$$= -(\lambda^3 - 21\lambda - 20) - 18\lambda - 18$$

$$= -(\lambda^3 - 21\lambda - 20 + 18\lambda + 18) = -(\lambda^3 - 3\lambda - 2)$$

$$= -(\lambda+1)(\lambda^2-\lambda-2) = -(\lambda+1)(\lambda-2)(\lambda+1)$$

$$\begin{array}{r} \lambda^2 - \lambda - 2 \\ \lambda+1 \overline{\big)\, \lambda^3 \quad - 3\lambda - 2} \\ \underline{\lambda^3 + \lambda^2} \\ -\lambda^2 - 3\lambda - 2 \\ \underline{-\lambda^2 - \lambda} \\ -2\lambda - 2 \\ \underline{-2\lambda - 2} \\ 0 \end{array}$$

$\underline{\lambda = -1}$ $\begin{bmatrix} 3 & 3 & -3 \\ -3 & -3 & 3 \\ -3 & -3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ x_2 free, x_3 free $\quad \vec{x} = \begin{bmatrix} -x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix}$

$$= x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 2}$: $\begin{bmatrix} 0 & 3 & -3 \\ -3 & -6 & 3 \\ -3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ -3 & -6 & 3 \\ 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

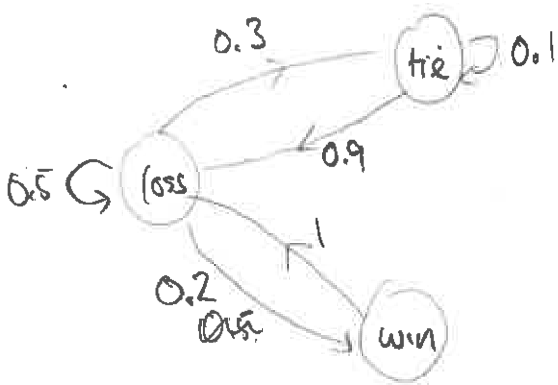
$$\vec{x} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v_3} = \quad x_3 \text{ free}$$

$$\boxed{\text{So } P = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ works.}}$$

~~20~~ 25

4. A casino operates a slot machine, where each play costs 1 dollar. The machine either returns no cash (a loss), returns 1 dollar (a tie), or returns 5 dollars (a win). The chances of an outcome depend on the previous outcome. If the previous outcome was a loss, then the next outcome has a 50% chance of being another loss, a 30% chance of being a tie, and a 20% chance of being a win. If the previous outcome was a tie, then the next outcome has a 90% chance of being a loss and a 10% chance of being another tie. If the previous outcome is a win, then the next outcome has a 100% chance of being a loss.

(a) [~~4~~ 5 **points**] Draw a state diagram that models the slot machine.



(b) [~~4~~ 5 **points**] Give the stochastic matrix P for the corresponding Markov chain.

$$P = \begin{array}{c} \\ \text{loss} \\ \text{tie} \\ \text{win} \end{array} \begin{array}{c} \begin{array}{ccc} \text{loss} & \text{tie} & \text{win} \end{array} \\ \begin{bmatrix} 0.5 & 0.9 & 1 \\ 0.3 & 0.1 & 0 \\ 0.2 & 0 & 0 \end{bmatrix} \end{array}$$

(c) [~~8~~ 10 **points**] Find the steady state vector for P.

$$\mathrm{Nul}(P-I): \begin{bmatrix} -0.5 & 0.9 & 1 \\ 0.3 & -0.9 & 0 \\ 0.2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} -5 & 9 & 10 \\ 3 & -9 & 0 \\ 1 & 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 \\ 3 & -9 & 0 \\ -5 & 9 & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & -9 & 15 \\ 0 & 9 & -15 \end{bmatrix} \to \begin{array}{c} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix} \\ x_1 \quad x_2 \quad \underset{\text{free}}{x_3} \end{array} \quad x = \begin{bmatrix} 5x_3 \\ 5/3\, x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ 5/3 \\ 1 \end{bmatrix}$$

Need $x_3\left(5 + \frac{5}{3} + 1\right) = 1$; $x_3(15+5+3) = 3$; $x_3 = \frac{3}{23}$

So $q = \frac{3}{23}\begin{bmatrix} 5 \\ 5/3 \\ 1 \end{bmatrix} = \boxed{\frac{1}{23}\begin{bmatrix} 15 \\ 5 \\ 3 \end{bmatrix}}$ loss, tie, win

(d) [~~4~~ 5 **points**] On average, how much money does the player lose on each play?

Net value: $\underbrace{\frac{15}{23}(-1)}_{\text{loss}} + \underbrace{\frac{5}{23}(0)}_{\text{tie}} + \underbrace{\frac{3}{23}(4)}_{\text{win}} = -\frac{15}{23} + \frac{12}{23} = \frac{-3}{23}$

So the loss is $\boxed{\frac{3}{23}}$ dollars, or $\frac{300}{23}$ $23\overline{)300}$ = 13.04 (23, 70, 69, 100) or about $\boxed{13.04}$ cents.

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5. [~~5~~ 6 parts, 2 points each] True/False. Justify your answer.

(a) If A has a strictly dominant eigenvalue λ, then for most vectors $\mathbf{x}_0$, the sequence $\mathbf{x}_0, \mathbf{x}_1, \ldots$ defined by $\mathbf{x}_k = A^k\mathbf{x}_0$ approaches an eigenvector for λ.

$\boxed{\text{FALSE}}$ Only the direction approaches an eigenvector

(b) If A and B are $n \times n$ matrices, then $\det(A+B) = \det(A) + \det(B)$.

$\boxed{\text{FALSE}}$ If $A = B = I_n$, then $\det(A+B) = 2^n$ but $\det(A) + \det(B) = 2$

(c) If A and B are similar matrices, then A and B have the same eigenvalues with the same multiplicities.

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$\boxed{\text{True}}$ The char. polynomial is the same

(d) Let A be a matrix with 7 rows and 10 columns. If the null space of A has dimension 4, then the column space of A has dimension 3.

$A \overbrace{[\quad]}^{10}$ $\boxed{\text{False}}$ The column space has dim $10-4$ or 6.

(e) If A is a square matrix and r is a scalar, then A is similar to rA.

$\boxed{\text{FALSE}}$ If $A = I_n$, ~~and~~ $r = 2$, then A and rA have char. polynomial $(1-\lambda)^n$ and $(2-\lambda)^n$ resp.

(f) If A is diagonalizable, then so is $A^2 + A$.

$\boxed{\text{TRUE}}$. $A = PDP^{-1} \Rightarrow A^2 + A = PD^2P^{-1} + PDP^{-1} = P(D^2+D)P^{-1}$.

6. Find the distance between the complex-valued vectors $\underset{u}{\begin{bmatrix} 3+i \\ 2 \end{bmatrix}}$ and $\underset{v}{\begin{bmatrix} -1+2i \\ i \end{bmatrix}}$.

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$u - v = \begin{bmatrix} 4-i \\ 2-i \end{bmatrix}$. $\text{dist}(u,v) = \|u-v\|^{1/2} = ((u-v)\cdot(u-v))^{1/2} = ((4-i)(4+i) + (2-i)(2+i))^{1/2}$

$= ((16 - i^2) + (4 - i^2))^{1/2} = (17+5)^{1/2} = \boxed{\sqrt{22}}$.

7. Find a unit vector in the direction of $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$.

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$w = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$ $\|w\| = \sqrt{6^2+3^2+2^2} = \sqrt{36+9+4} = \sqrt{49} = 7$

So unit vector is $\begin{bmatrix} 6/7 \\ 3/7 \\ 2/7 \end{bmatrix}$.