Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[15 points]** Solve the following system of equations.

$$\begin{array}{rcrcrcr} 3x_1 & - & x_2 & + & 2x_3 & = & 12 \\ x_1 & + & x_2 & - & x_3 & = & -2 \\ 4x_1 & - & 5x_2 & + & 2x_3 & = & 1 \end{array}$$

$$\begin{bmatrix} 3 & -1 & 2 & 12 \\ 1 & 1 & -1 & -2 \\ 4 & -5 & 2 & 1 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 3 & -1 & 2 & 12 \\ 4 & -5 & 2 & 1 \end{bmatrix} \xrightarrow[R3 \pm (-4)R1]{R2 \pm (-3)R1} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -4 & 5 & 18 \\ 0 & -9 & 6 & 9 \end{bmatrix} \xrightarrow[R3 \cdot (\frac{1}{3})]{R2 \cdot (-1)} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 4 & -5 & -18 \\ 0 & -3 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{R2 \pm R3} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & -3 & -15 \\ 0 & -3 & 2 & 3 \end{bmatrix} \xrightarrow{R3 \pm 3R2} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & -3 & -15 \\ 0 & 0 & -7 & -42 \end{bmatrix} \xrightarrow{R3 \cdot (-\frac{1}{7})} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & -3 & -15 \\ 0 & 0 & 1 & 6 \end{bmatrix} \xrightarrow[R2 \pm 3(R3)]{R1 \pm R3} \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

$$\xrightarrow{R1 \pm (-1)R2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{bmatrix} \qquad \boxed{\begin{array}{l} x_1 = 1 \\ x_2 = 3 \\ x_3 = 6 \end{array}}$$

2. **[5 parts, 2 points each]** True/False. If the statement is true, then explain why. If the statement is false, then edit the statement slightly to make it true.

   (a) In a matrix with 10 rows, replacing row 8 with $2(R7) - R8$ (i.e. 2 times row 7 minus row 8) is an elementary row operation.

   (b) Let A be a matrix and let A' be a matrix obtained from A by swapping two columns. The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $A'\mathbf{x} = \mathbf{b}$ is consistent.

   (c) Every inconsistent system is nonhomogeneous.

   (d) The solution set for $A\mathbf{x} = \mathbf{b}$ is a translation of the solution set for $A\mathbf{x} = \mathbf{0}$.

   (e) Every matrix is row equivalent to a unique matrix in reduced echelon form.

(a) FALSE. $R8 \leftarrow R8 + 2(R7)$ is an elementary row operation.

(b) True. Suppose we obtain A' by swapping the i^{th} and j^{th} columns.

If $\vec{x}$ is a soln to $A\vec{x} = \vec{b}$, then the vector $\vec{x}'$ obtained from $\vec{x}$ by swapping the i^{th} and j^{th} entry of $\vec{x}$ is a soln to $A'\vec{x}' = \vec{b}$, and vice versa.

(c) True. A homogeneous system $A\vec{x} = \vec{0}$ always has the solution $\vec{x} = \vec{0}$ (trivial soln) and therefore every homogeneous system is consistent. So the only inconsistent systems are non homogeneous.

(d) False. The solution set for a consistent system $A\vec{x} = \vec{b}$ is a translation of the solution set for $A\vec{x} = \vec{0}$.

(e) True. This is an important fact from Section 1.2.

3. **[10 points]** Determine the value(s) of h, if any, make the linear system represented by the following augmented matrix consistent.

$$\begin{bmatrix} 6 & -1 & 1 & 7 \\ 2 & h & -2 & -10 \\ 1 & 2 & 3 & 7 \\ 3 & 0 & 1 & 5 \end{bmatrix}$$

Re order Rows:

$$\begin{bmatrix} 1 & 2 & 3 & 7 \\ 6 & -1 & 1 & 7 \\ 3 & 0 & 1 & 5 \\ 2 & h & -2 & -10 \end{bmatrix} \xrightarrow[\substack{R3 \pm (-3)R1 \\ R4 \pm (-2)R1}]{R2 \pm (-6)R1} \begin{bmatrix} 1 & 2 & 3 & 7 \\ 0 & -13 & -17 & -35 \\ 0 & -6 & -8 & -16 \\ 0 & h-4 & -8 & -24 \end{bmatrix} \xrightarrow{R2 \cdot (-1)} \begin{bmatrix} 1 & 2 & 3 & 7 \\ 0 & 13 & 17 & 35 \\ 0 & -6 & -8 & -16 \\ 0 & h-4 & -8 & -24 \end{bmatrix} \xrightarrow[R3 \cdot (\frac{1}{2})]{R2 \pm 2R3} \begin{bmatrix} 1 & 2 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & -3 & -4 & -8 \\ 0 & h-4 & -8 & -24 \end{bmatrix}$$

$$\xrightarrow[R4 \pm (4-h)R2]{R3 \pm 3R2} \begin{bmatrix} 1 & 2 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -4-h & -24+3(4-h) \end{bmatrix} \xrightarrow[R4 \cdot (-1)]{R3 \cdot (-1)} \begin{bmatrix} 1 & 2 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 4+h & 12+3h \end{bmatrix} \xrightarrow{R4 \pm -(4+h)R3} \begin{bmatrix} 1 & 2 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 16+4h \end{bmatrix}$$

$-24+12-3h$

For system to be consistent, we need $16+4h=0$, or $\boxed{h=-4}$.

4. **[15 points]** Give an equation for the components of $\mathbf{b}$ that determines when the system $A\mathbf{x} = \mathbf{b}$ is consistent.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 8 & -4 & 12 \\ 1 & -2 & 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & b_1 \\ 8 & -4 & 12 & b_2 \\ 1 & -2 & 3 & b_3 \end{bmatrix} \xrightarrow[R3 \pm -R1]{R2 \pm -8R1} \begin{bmatrix} 1 & 2 & -1 & b_1 \\ 0 & -20 & 20 & b_2 - 8b_1 \\ 0 & -4 & 4 & b_3 - b_1 \end{bmatrix} \xrightarrow{R3 \cdot (-5)} \begin{bmatrix} 1 & 2 & -1 & b_1 \\ 0 & -20 & 20 & b_2 - 8b_1 \\ 0 & 20 & -20 & -5b_3 + 5b_1 \end{bmatrix}$$

$$\xrightarrow{R3 \pm R2} \begin{bmatrix} 1 & 2 & -1 & b_1 \\ 0 & -20 & 20 & b_2 - 8b_1 \\ 0 & 0 & 0 & -5b_3 + 5b_1 + b_2 - 8b_1 \end{bmatrix}$$

We need $5b_3 + 5b_1 + b_2 - 8b_1 = 0$

$\boxed{-3b_1 + b_2 - 5b_3 = 0}$ or $\boxed{3b_1 - b_2 + 5b_3 = 0}$

5. **[10 points]** Suppose that $\mathbf{v_1}, \ldots, \mathbf{v_n}$ are vectors in $\mathbb{R}^m$ that span $\mathbb{R}^m$. Give a careful argument that $n \geq m$.

Since $\text{Span}(\{v_1, \ldots, v_n\}) = \mathbb{R}^m$, we have that for each $\vec{b} \in \mathbb{R}^m$, there is a solution to $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{b}$. Let $A = [\vec{v}_1 \cdots \vec{v}_n]$, so that A is the coefficient matrix for this system. For a particular $\vec{b} \in \mathbb{R}^m$, the system $A\vec{x} = \vec{b}$ has a soln iff the augmented matrix $[\vec{v}_1 \cdots \vec{v}_n\ \vec{b}]$ has no pivot in the last column. For $A\vec{x} = \vec{b}$ to be consistent for <u>each</u> $\vec{b} \in \mathbb{R}^m$, we must have a pivot entry in each row of A. Since each column of A has at most one pivot entry, we have #rows of $A \leq$ #cols of A. Since A has m rows and n columns, it follows that $m \leq n$.

6. **[10 points]** Besides $\mathbf{0}$, is there a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ that has no negative entries? Justify your answer.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} \qquad \mathbf{a}_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \qquad \mathbf{a}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

We determine the vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ that are linear combinations of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$.

$$\begin{bmatrix} 1 & 2 & -2 & b_1 \\ 3 & -3 & 1 & b_2 \\ -4 & 1 & 1 & b_3 \end{bmatrix} \xrightarrow[R3 \pm (4)R1]{R2 \pm (-3)R1} \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & -9 & 7 & b_2 - 3b_1 \\ 0 & 9 & -7 & b_3 + 4b_1 \end{bmatrix} \xrightarrow{R3 \pm R2} \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & -9 & 7 & b_2 - 3b_1 \\ 0 & 0 & 0 & b_3 + 4b_1 + b_2 - 3b_1 \end{bmatrix}$$

The vector $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is a lin. comb. iff the system whose augmented matrix appears above is consistent iff $b_3 + 4b_1 + b_2 - 3b_1 = 0$ iff $b_1 + b_2 + b_3 = 0$.

The ~~[illegible]~~ linear combinations of $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are the vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1 + b_2 + b_3 = 0$. So if there are no negative entries, then all entries are zero. So $\boxed{\text{No,}}$ there is no other linear combination with no negative entries.

7. **[15 points]** Express the solution set to $A\mathbf{x} = \mathbf{0}$ in parametric form.

$$A = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 & -3 & -2 & 0 \\ 0 & 0 & 1 & 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$x_7 = 0$

x_5, x_6 free

$x_4 = -x_5 + x_6$

$x_3 = -4x_5 - 3x_6$

x_2 free

$x_1 = -2x_2 + 3x_5 + 2x_6$

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ -4 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_2, x_5, x_6 \text{ free in } \mathbb{R}.$$

8. **[15 points]** Express the solution set to $A\mathbf{x} = \mathbf{b}$ in parametric form.

$$A = \begin{bmatrix} 5 & 20 & -5 & -2 & 1 \\ 2 & 8 & -2 & 1 & -1 \\ -3 & -12 & 3 & 0 & -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 12 \\ 8 \\ -8 \end{bmatrix}$$

Augmented matrix:

$$\begin{bmatrix} 5 & 20 & -5 & -2 & 1 & 12 \\ 2 & 8 & -2 & 1 & -1 & 8 \\ -3 & -12 & 3 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{R1 \pm (-2)R2} \begin{bmatrix} 1 & 4 & -1 & -4 & 3 & -4 \\ 2 & 8 & -2 & 1 & -1 & 8 \\ -3 & -12 & 3 & 0 & -1 & -8 \end{bmatrix} \xrightarrow[R3 \pm (3)R1]{R2 \pm (-2)R1} \begin{bmatrix} 1 & 4 & -1 & -4 & 3 & -4 \\ 0 & 0 & 0 & 9 & -7 & 16 \\ 0 & 0 & 0 & -12 & 8 & -20 \end{bmatrix}$$

$$\xrightarrow{R3 \cdot (-\frac{1}{4})} \begin{bmatrix} 1 & 4 & -1 & -4 & 3 & -4 \\ 0 & 0 & 0 & 9 & -7 & 16 \\ 0 & 0 & 0 & 3 & -2 & 5 \end{bmatrix} \xrightarrow{R3 \pm (-3)R2} \begin{bmatrix} 1 & 4 & -1 & -4 & 3 & -4 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 & -2 & 5 \end{bmatrix} \xrightarrow[R2 \leftrightarrow R3]{R2 \cdot (-1)} \begin{bmatrix} 1 & 4 & -1 & -4 & 3 & -4 \\ 0 & 0 & 0 & 3 & -2 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow[R2 \pm (2)R3]{R1 \pm (-3)R3} \begin{bmatrix} 1 & 4 & -1 & -4 & 0 & -1 \\ 0 & 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R2 \cdot \frac{1}{3}} \begin{bmatrix} 1 & 4 & -1 & -4 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R1 \pm 4(R2)} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 4 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \end{array}$$

$x_5 = -1$

$x_4 = 1$

x_2, x_3 free

$x_1 = 3 - 4x_2 + x_3$

$$\Rightarrow \quad \vec{x} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_2, x_3 \in \mathbb{R}.$$