Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[2 parts, 2 points each]** Decide whether the given transformation is linear. Justify your answer.

(a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1+1 \\ x_2-1 \end{bmatrix}$

This transform maps $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0+1 \\ 0-1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Every linear transform maps the zero vector to the zero vector, so this is $\boxed{\text{not linear.}}$

(b) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} \ln(2)x_1 \\ -x_2 \end{bmatrix}$.

This is linear: $T(\vec{x}) = \begin{bmatrix} \ln(2)\, x_1 \\ -x_2 \end{bmatrix} = \begin{bmatrix} \ln(2)\, x_1 + 0x_2 \\ 0x_1 + (-1)x_2 \end{bmatrix}$

$$= \begin{bmatrix} \ln(2) & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Since the transform is a matrix transform with standard matrix $\begin{bmatrix} \ln(2) & 0 \\ 0 & -1 \end{bmatrix}$, $\boxed{\text{It is linear.}}$

2. **[1 point]** Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transform, and let $\mathbf{v}_1, \ldots \mathbf{v}_p$ be vectors in $\mathbb{R}^n$. Show that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a linearly dependent set, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is linearly dependent.

Pf. Since $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is lin. dependent, there exist weights $w_1, \ldots, w_p$, not all zero, such that $w_1\vec{v}_1 + \cdots + w_p\vec{v}_p = \vec{0}$. Since T is linear, we have that

$$w_1 T(\vec{v}_1) + \cdots + w_p T(\vec{v}_p) = T(w_1\vec{v}_1 + \cdots + w_p\vec{v}_p) = T(\vec{0}) = \vec{0}.$$

It follows that $w_1 T(\vec{v}_1) + \cdots + w_p T(\vec{v}_p) = \vec{0}$ is a dependence relation for $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$, and so $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ is linearly dependent. $\boxtimes$

3. **[2 parts, 2 points each]** Suppose that $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transform, let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and let $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. We know that T maps $\mathbf{u}$ to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and T maps $\mathbf{v}$ to $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

   (a) Find the image of $2\mathbf{u} - \mathbf{v}$ under T.

$$T(2\vec{u} - \vec{v}) = 2T(\vec{u}) - T(\vec{v}) = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - (-1) \\ 4 - 0 \\ 6 - 2 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}}.$$

   (b) If possible, then find $T(\mathbf{w})$, where $\mathbf{w} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$. If not possible, then explain why not.

First, try to write $\vec{w}$ as a lin. comb. of $\vec{u}$ and $\vec{v}$: $x_1\vec{u} + x_2\vec{v} = \vec{w} \Leftrightarrow \begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{w}$

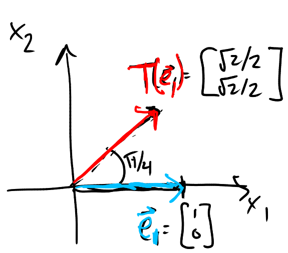
Augmented matrix: $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & -1 & 8 \end{array}\right] \to \begin{bmatrix} 1 & 2 & 5 \\ 0 & -7 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

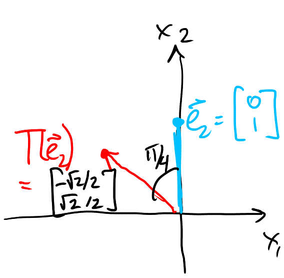
So $\vec{w} = 3\vec{u} + 1\vec{v}$. (Check: $\begin{bmatrix} 5 \\ 8 \end{bmatrix} \overset{?}{=} 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ ✓).

Therefore $T(\vec{w}) = T(3\vec{u} + \vec{v}) = 3T(\vec{u}) + T(\vec{v}) = 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 - 1 \\ 6 + 0 \\ 9 + 2 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 6 \\ 11 \end{bmatrix}}.$

4. **[1 point]** Give the standard matrix for the transform $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates the plane by 45 degrees counter-clockwise.

Std matrix is $\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \sqrt{2}/2 & \frac{\sqrt{2}}{2} \end{bmatrix} = \boxed{\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}$





Alt soln, from formula in class:

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} = \begin{bmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$