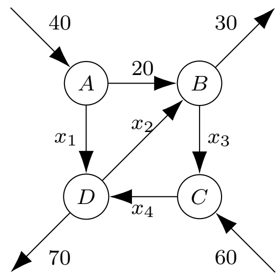
Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[2.5 points]** Consider the following network.



Express the general solution to this network flow in parametric form.

A: $40 = 20 + x_1 \Rightarrow x_1 = 20$

B: $x_2 + 20 = x_3 + 30 \Rightarrow x_2 - x_3 = 10$

C: $x_3 + 60 = x_4 \Rightarrow -x_3 + x_4 = 60$

D: $x_1 + x_4 = x_2 + 70 \Rightarrow x_1 - x_2 + x_4 = 70$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & -1 & 0 & 10 \\ 0 & 0 & -1 & 1 & 60 \\ 1 & -1 & 0 & 1 & 70 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & -1 & 0 & 10 \\ 0 & 0 & -1 & 1 & 60 \\ 0 & -1 & 0 & 1 & 50 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & -1 & 0 & 10 \\ 0 & 0 & -1 & 1 & 60 \\ 0 & 0 & -1 & 1 & 60 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & -1 & 0 & 10 \\ 0 & 0 & -1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & -1 & -50 \\ 0 & 0 & 1 & -1 & -60 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 20$
$x_2 = -50 + x_4$
$x_3 = -60 + x_4$
x_4: free

$$x = \begin{bmatrix} 20 \\ -50 + x_4 \\ -60 + x_4 \\ x_4 \end{bmatrix} = \boxed{\begin{bmatrix} 20 \\ -50 \\ -60 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad s \in \mathbb{R}}$$

2. **[2.5 points]** Determine (with justification) the set of values for h that make $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6\}$ linearly dependent, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 8 \\ 5 \\ -1 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ h \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 7 \\ 3 \\ -1 \\ 2 \\ 6 \end{bmatrix} \quad \mathbf{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{u}_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{u}_6 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}.$$

Since $\{\vec{u}_1, \ldots, \vec{u}_6\}$ is a collection of 6 vectors in $\mathbb{R}^5$ and $6 > 5$, this set is $\boxed{\text{linearly dependent for all } h}$. Indeed, the matrix A with $A = [\vec{u}_1 \cdots \vec{u}_6]$ has 6 columns and 5 rows. So at least one column is not a pivot column, implying the homogeneous system $A\vec{x} = \vec{0}$ has a free variable and therefore nontrivial solutions.

3. **[2.5 points]** Determine (with justification) whether the columns of the following matrix are linearly dependent.

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & 10 \\ 4 & 1 & 10 \\ -1 & 3 & -9 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & 10 \\ 4 & 1 & 10 \\ -1 & 3 & -9 \end{bmatrix} \xrightarrow[\substack{R3 \mathrel{+}= -4R1 \\ R4 \mathrel{+}= R1}]{R2 \mathrel{+}= -2R1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -6 & 12 \\ 0 & -7 & 14 \\ 0 & 5 & -10 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the third column is not a pivot column, the columns are $\boxed{\text{linearly dependent}}$.

Check work (not required): $\rightsquigarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_3 = 1$, $x_2 = 2$, $x_1 = -3$

$$-3\begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -2 \\ 1 \\ 3 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 10 \\ 10 \\ -9 \end{bmatrix} = \vec{0} \checkmark$$

4. **[2.5 points]** Is there a linearly dependent set of vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ such that $\{\mathbf{u}, \mathbf{v}\}$, $\{\mathbf{u}, \mathbf{w}\}$, and $\{\mathbf{v}, \mathbf{w}\}$ are all linearly independent? Give an example or explain why not.

$\boxed{\text{Yes.}}$ For example, $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Since $\{u, v, w\}$ are 3 vectors, each with 2 entries, $\{u, v, w\}$ is linearly dependent. Since no vector is a scalar multiple of another, the pairs are linearly independent.