Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[2 points]** Given **y** and **v** below, decompose **y** as $\mathbf{y} = c\mathbf{v} + \mathbf{z}$ where c is a scalar and $\mathbf{z} \cdot \mathbf{v} = \mathbf{0}$.

$$\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

$$c = \frac{y \cdot v}{v \cdot v} = \frac{(1)(-1) + (2)(3) + (3)(1)}{(-1)^2 + 3^2 + 1^2} = \frac{-1+6+3}{1+9+1} = \frac{8}{11}, \quad z = y - cv = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{8}{11}\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 11 \\ 22 \\ 33 \end{bmatrix} - \frac{1}{11}\begin{bmatrix} -8 \\ 24 \\ 8 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 19 \\ -2 \\ 25 \end{bmatrix}$$

So $\vec{y} = \boxed{\frac{-8}{11}\vec{v} + \frac{1}{11}\begin{bmatrix} 19 \\ -2 \\ 25 \end{bmatrix}}$

2. **[2 parts, 2 points each]** Define $\mathbf{v}_1, \mathbf{v}_2, \mathbf{y}$ as follows and let $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 3 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

(a) Find $\text{proj}_W(\mathbf{y})$. Note: $v_1 \cdot v_2 = (-2) + (-2) + 1 + 3 = 0$ so $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for W.

$$\text{proj}_W y = c_1 v_1 + c_2 v_2, \quad c_j = \frac{y \cdot v_j}{v_j \cdot v_j}. \quad c_1 = \frac{-2+2+9}{4+4+1+9} = \frac{9}{18} = \frac{1}{2}, \quad c_2 = \frac{1+2+3}{1+1+1+1} = \frac{6}{4} = \frac{3}{2}$$

$$\text{So } y = \frac{1}{2}\begin{bmatrix} 2 \\ -2 \\ 1 \\ 3 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2-3 \\ -2+3 \\ 1+3 \\ 3+3 \end{bmatrix} = \boxed{\frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 4 \\ 6 \end{bmatrix}}$$

(b) Find the distance from **y** to W.

$$\text{dist}(y, W) = \|z\|, \text{ where } z = y - \hat{y} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 4 \\ 6 \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 4 \\ 6 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{So } \|z\| = (z \cdot z)^{1/2} = \left(\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + 0^2 + 0^2\right)^{1/2} = \left(\tfrac{1}{4} + \tfrac{1}{4} + 0\right)^{1/2} = \left(\tfrac{1}{2}\right)^{1/2}$$

$$= \frac{1}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{2}}$$

3. **[1 point]** Let W be a subspace of $\mathbb{R}^n$. Suppose that $\mathbf{y} = \hat{\mathbf{y}}_1 + \mathbf{z}_1 = \hat{\mathbf{y}}_2 + \mathbf{z}_2$ where $\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2 \in W$ and $\mathbf{z}_1, \mathbf{z}_2 \in W^\perp$. Prove that $\hat{\mathbf{y}}_1 = \hat{\mathbf{y}}_2$ and $\mathbf{z}_1 = \mathbf{z}_2$. [Hint: begin with $\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2 = \mathbf{z}_2 - \mathbf{z}_1$ and take the dot product with an appropriate vector on both sides.]

Since W is a subspace and $\hat{y}_1, \hat{y}_2 \in W$, we have $\hat{y}_1 - \hat{y}_2 \in W$. So

$$\hat{y}_1 - \hat{y}_2 = z_2 - z_1 \Rightarrow (\hat{y}_1 - \hat{y}_2)\cdot(\hat{y}_1 - \hat{y}_2) = (z_2 - z_1)\cdot(\hat{y}_1 - \hat{y}_2)$$
$$= z_2 \cdot (\hat{y}_1 - \hat{y}_2) - z_1 \cdot (\hat{y}_1 - \hat{y}_2)$$
$$= 0 - 0 = 0,$$

since z_1 and z_2 are orthogonal to each vector in W. So $\|\hat{y}_1 - \hat{y}_2\| = ((\hat{y}_1 - \hat{y}_2)\cdot(\hat{y}_1 - \hat{y}_2))^{1/2} = 0$ which implies $\hat{y}_1 = \hat{y}_2$. So $z_2 - z_1 = \hat{y}_1 - \hat{y}_2 = \vec{0}$ and therefore $z_1 = z_2$ also.

4. **[3 parts, 1 point each]** True/False. In the following, A and B are $n \times n$ matrices. Justify your answer.

   (a) If A has orthogonal columns, then A^T also has orthogonal columns.

$\boxed{\text{False.}}$ For example, if $A = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$ then A has orthogonal columns $\left((1)(2) + (1)(-2) = 0\right)$

but $A^T = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$ and so the columns of A^T are not orthogonal

$\left((1)(1) + (2)(-2) = -3 \neq 0\right)$.

   (b) A has orthonormal columns if and only if $A^T A = I$.

$\boxed{\text{True}}$: Suppose $A = [v_1 \cdots v_n]$. Then the entry of $A^T A$ in the i^{th} row and j^{th} column is $v_i \cdot v_j$. When $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal set, $v_i \cdot v_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$ and so $A^T A = I$. Conversely, if $A^T A = I$, then $v_i \cdot v_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$, which implies $\{v_1, \dots, v_n\}$ is an orthonormal set.

   (c) If A and B have orthonormal columns, then so does AB.

$\boxed{\text{True}}$. Since A and B have orthonormal columns, we have $A^T A = I$ and $B^T B = I$.

So $(AB)^T(AB) = B^T A^T A B = B^T I B = B^T B = I$. Therefore AB also has orthonormal columns.