Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[4 parts, 1 point each]** True/False. In the following, A and B are $n \times n$ matrices. Justify your answers.

   (a) For $n \geq 4$, an $(n \times n)$-matrix has at least 2 linearly independent eigenvectors.

   False. For example, $\begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{bmatrix}$ has value $\lambda = 0$ (mult. 0), but the eigenspace for $\lambda = 0$ is 1-dimensional $\left(= \text{Span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right\}\right)$. So no 2 lin. indep. eigenvalues.

   (b) The char. polynomial of A has enough information to tell whether A is diagonalizable.

   FALSE. For example $\begin{bmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{bmatrix}$ is not diagonalizable and the all zeros matrix is, but both have the same characteristic polynomial $f(\lambda) = (0-\lambda)^4 = \lambda^4$.

   (c) If A and B are similar and A is diagonalizable, then B is also diagonalizable.

   True. If A is diagonalizable, then A is similar to a diagonal matrix D. Since B and A are similar and similarity is transitive, B is also similar to D and therefore diagonalizable. ∎

   (d) If A and B are similar matrices, then $\det(A) = \det(B)$.

   True. If A and B are similar, then $A = P^{-1}BP$ for some matrix P. So
   $\det(A) = \det(P^{-1}BP) = \det(P^{-1}) \cdot \det(B) \cdot \det(P) = \det(B) \cdot \det(P^{-1} \cdot P) = \det(B) \cdot \det(I)$
   $= \det(B) \cdot 1 = \det(B)$. ∎

2. **[2 parts, 3 points each]** Diagonalize the following matrices if possible. That is, for each diagonalizable matrix A below, construct an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. (There is no need to compute P^{-1} explicitly.) For each matrix A below that is not diagonalizable, explain why not.

   (a) $\begin{bmatrix} -15 & -14 \\ 21 & 20 \end{bmatrix}$

   $$0 = \begin{vmatrix} -15-\lambda & -14 \\ 21 & 20-\lambda \end{vmatrix} = (-15-\lambda)(20-\lambda) - (-14)(21) = (\lambda^2 - 5\lambda - 3.5 \cdot 4.5) + 2 \cdot 7 \cdot 3 \cdot 7$$
   $$= \lambda^2 - 5\lambda + 6 \cdot 7^2 - 6 \cdot 2.5^2 = \lambda^2 - 5\lambda + 6(49-50)$$
   $$= \lambda^2 - 5\lambda - 6 = (\lambda-6)(\lambda+1). \text{ So } \lambda_1 = 6, \lambda_2 = -1$$

   $\lambda_1 = 6$: $\begin{bmatrix} -21 & -14 \\ 21 & 14 \end{bmatrix} \sim \begin{bmatrix} 21 & 14 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

   $\lambda_2 = -1$: $\begin{bmatrix} -14 & -14 \\ 21 & 21 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   So $A = PDP^{-1}$ where $P = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$.

   Check: $PDP^{-1} = \begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix} \left(\frac{1}{(-2)+3}\right) \begin{bmatrix} -1 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} -6 & -6 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -15 & -14 \\ 21 & 20 \end{bmatrix}$ ✓.

(b) $\begin{bmatrix} -7 & 0 & 10 \\ -10 & 3 & 10 \\ -8 & 0 & 11 \end{bmatrix}$ $0 = \begin{vmatrix} -7-\lambda & 0 & 10 \\ -10 & 3-\lambda & 10 \\ -8 & 0 & 11-\lambda \end{vmatrix} = (\lambda-3)\begin{vmatrix} -7-\lambda & 10 \\ -8 & 11-\lambda \end{vmatrix} = (\lambda-3)\left[(-7-\lambda)(11-\lambda) + 8\cdot 10\right]$

$= (\lambda-3)\left[\lambda^2 - 4\lambda - 77 + 80\right] = (\lambda-3)\left[\lambda^2 - 4\lambda + 3\right] = (\lambda-3)(\lambda-3)(\lambda-1)$, $\lambda_1 = \lambda_2 = 3$, $\lambda_3 = 1$

$\underline{\lambda_1 = \lambda_2 = 3}$: $\begin{bmatrix} -10 & 0 & 10 \\ -10 & 0 & 10 \\ -8 & 0 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (x_1, x_2 x_3 free) $\vec{x} = \begin{bmatrix} x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

So $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$\underline{\lambda_3 = 1}$: $\begin{bmatrix} -8 & 0 & 10 \\ -10 & 2 & 10 \\ -8 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} -4 & 0 & 5 \\ -5 & 1 & 5 \\ -4 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ -5 & 1 & 5 \\ -4 & 0 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & -4 & 5 \\ 0 & -4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & -4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & -5/4 \\ 0 & 1 & -5/4 \\ 0 & 0 & 0 \end{bmatrix}$ (x_3 free) $\vec{x} = \begin{bmatrix} 5/4\, x_3 \\ 5/4\, x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 5/4 \\ 5/4 \\ 1 \end{bmatrix} = \hat{x}_3\begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$, so $\vec{v}_3 = \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$.

So $A = PDP^{-1}$ where $P = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = \begin{bmatrix} 0 & 1 & 5 \\ 1 & 0 & 5 \\ 0 & 1 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.