Directions: You may work to solve these problems in groups, but all written work must be your own. Show all work; no credit for solutions without work.

1. [6.1.24] Verify the *parallelogram law* for $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$: $||\mathbf{u}+\mathbf{v}||^2+||\mathbf{u}-\mathbf{v}||^2 = 2||\mathbf{u}||^2+2||\mathbf{v}||^2$.

2. [6.1.29] Let $W = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Show that if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$, then $\mathbf{x}$ is orthogonal to every vector in W.

3. [6.2.9] Define vectors as follows.

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \qquad x = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$$

   (a) Show that $\{u_1, u_2, u_3\}$ is an orthogonal set.

   (b) Express x as a linear combination of u_1, u_2, and u_3.

4. [6.2.14] Let $\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of two orthogonal vectors, one in $\text{Span}\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

5. [6.2.15] Let $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

6. [6.2.25] Let U be an $m \times n$-matrix with orthonormal columns. Prove that $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

7. [6.3.12] Find the closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, and then find the distance from $\mathbf{y}$ to W.

$$\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix} \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

8. [6.3.21] True/False. All vectors and subspaces are in $\mathbb{R}^n$. Justify each answer.

   (a) If $\mathbf{z}$ is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$ and if $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, then $\mathbf{z} \in W^\perp$.

   (b) For each $\mathbf{y}$ and each subspace W, the vector $\mathbf{y} - \text{proj}_W \mathbf{y}$ is orthogonal to W.

   (c) The orthogonal projection $\hat{y}$ of y onto a subspace W can sometimes depend on the orthogonal basis for W used to compute $\hat{y}$.

   (d) If $\mathbf{y}$ is in a subspace W, then the orthogonal projection of $\mathbf{y}$ onto W is $\mathbf{y}$ itself.

   (e) If the columns of an $n \times p$ matrix U are orthonormal, then $UU^T\mathbf{y}$ is the orthonormal projection of $\mathbf{y}$ onto the column space of U.