Directions: Solve the following problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. Let M be the set of all positive integers m such that $a^{\phi(m)+1} \equiv a \pmod{m}$ for each integer a. Give a simple characterization of M and prove that your characterization is correct.

2. [NT 5-2.{9,10}]
   (a) Prove that if p is a prime and $p \equiv 1 \pmod{4}$, then $\left[\left(\frac{p-1}{2}\right)!\right]^2 \equiv -1 \pmod{p}$.
   (b) Use the above to find a solution for each of the following.
      i. $x^2 \equiv -1 \pmod{13}$
      ii. $x^2 \equiv -1 \pmod{17}$

3. Prove that if m is not a prime and $\phi(m) \mid m-1$, then m has at least three distinct prime factors.

4. Prove that the number of ways of writing n as a sum of consecutive positive integers equals $d(m)$, where m is the largest odd divisor of n. For example, for $n = 42$ we have $m = 21$ and $d(m) = d(21) = 4$. As expected, there are 4 ways to express 42 as the sum of consecutive positive integers: 42, $13 + 14 + 15$, $9 + 10 + 11 + 12$, and $3 + 4 + \cdots + 9$.

5. [NT 6-4.2] Prove that if $f(n)$ is multiplicative and not identically zero, then $\sum_{d \mid n} \mu(d) f(d) = \prod_{p \mid n} (1 - f(p))$.

6. Dirichlet product and Mobius pairs. Let $\mathbb{O}\colon \mathbb{Z}^+ \to \mathbb{R}$ be the identically zero function.
   (a) Prove that the Dirichlet product is bilinear. That is, for all functions $f, g, h\colon \mathbb{Z}^+ \to \mathbb{R}$ and all constants $\alpha, \beta \in \mathbb{R}$, we have $f * (\alpha g + \beta h) = \alpha(f * g) + \beta(f * h)$.
   (b) Prove that if $f * g = \mathbb{O}$, then $f = \mathbb{O}$ or $g = \mathbb{O}$.
   (c) Show that if both (h_1, h_2) and (h_2, h_1) are Mobius pairs, then $h_1 = h_2 = \mathbb{O}$.

7. **[Challenge]** Prove that if n divides $3^n - 1$, then $n = 1$ or n is even.