Directions: Solve the following problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. Prove that p is the smallest prime that divides $(p-1)!+1$.

2. [NT 5-2.4] Let $k=\phi(m)$. Prove that if $r_1,\ldots,r_k$ is a reduced residue system modulo m and m is odd, then $r_1+r_2+\cdots+r_k \equiv 0 \pmod{m}$.

3. [NT 5-3.4]

   (a) Prove that for each n, there are n consecutive integers, each of which is divisible by a perfect square larger than 1.

   (b) Using your proof above, explicitly find 3 consecutive integers, each of which is divisible by a perfect square larger than 1. In your answer, give the integers as well as the corresponding perfect squares.

4. [NT 5-4.1] Find the set of solutions to the following system of congruences:

$$\begin{aligned}
2x &\equiv 1 \pmod{5}\\
3x &\equiv 9 \pmod{6}\\
4x &\equiv 1 \pmod{7}\\
5x &\equiv 9 \pmod{11}
\end{aligned}$$

5. Let $A=\{a^2-b^2\colon\ a,b\in\mathbb{Z}\}$. Give a simple characterization of A (with proof of correctness).

6. Prove that there are infinitely many integers n such that $n \mid 2^n+1$. Hint: first, find the three smallest such integers.

7. **[Challenge]** Prove that if $n\geq 2$, then the sum $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ is not an integer.