Directions: Solve the following problems. Your solutions should be electronically typeset and all written work should be your own.

1. *Sums of squares.*
   (a) [NT 1-1.1] Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
   (b) Determine the set A of positive integers n such that n divides the sum $1^2 + 2^2 + \cdots + n^2$. Note: your answer should give a simple description of A and a proof that n divides $1^2 + \cdots + n^2$ if and only if $n \in A$.

2. Prove that for each non-negative integer n, we have that 169 divides $3^{3n+3} - 26n - 27$.

3. *Divisibility and exponents.*
   (a) Prove that if a and n are integers with $a \geq 3$ and $n \geq 2$, then $a^n - 1$ is not prime.
   (b) Prove that if $2^n - 1$ is prime, then n is prime.

4. Give the base 6 representation for 42,201.

5. Let $d = \gcd(63119, 38227)$. Find d and obtain integers p and q such that $d = 63119p + 38227q$. Show your work.

6. **[Challenge]** Let $A_n = \{(x, y)\colon\ 1 \leq x \leq n,\ 1 \leq y \leq n, \text{ and } \gcd(x, y) = 1\}$. Note that A_n contains all points (x, y) in the $(n \times n)$-grid with corners $(1, 1)$ and (n, n) such that the line segment joining $(0, 0)$ and (x, y) contains no other integer lattice points. The first few such sets are as follows:

$$A_1 = \{(1,1)\}$$
$$A_2 = \{(1,1),(2,1),(1,2)\}$$
$$A_3 = \{(1,1),(1,2),(1,3),(2,3),(2,1),(3,1),(3,2)\}$$

   Let $f(n) = |A_n|$. Note that $f(1) = 1$, $f(2) = 3$, $f(3) = 7$, and $f(4) = 11$. Prove that there is a positive constant C such that $f(n) \geq Cn^2$.