Name: Solutions

Directions: Show all work. Answers without work generally do not earn points. Unless stated otherwise, answers may be left in terms of factorials and binomial coefficients. Your answers should not involve sums with more than a few terms.

1. **[10 parts, 2 points each]** Let $A = \{\varnothing, \{1,2\}, \{1\}, \{2\}\}$, $B = \{2,1\}$, and $C = \{1, \{1\}, \{2\}\}$. For True/False questions, **write the whole word**. No work/explanation necessary.

(a) Determine $|A|$, $|B|$, and $|C|$.

$|A| = 4$

$|B| = 2$

$|C| = 3$ (Note $1 \neq \{1\}$)

(b) Determine $A \cap B$.

Members of A: sets

members of B: integers

$A \cap B = \boxed{\varnothing}$

(c) Determine $B \cup C$.

$B \cup C = \boxed{\{1, 2, \{1\}, \{2\}\}}$

(d) Determine $C - B$.

$C - B = \boxed{\{\{1\}, \{2\}\}}$

(e) Determine $A \triangle C$.

In exactly one of A, C

$A \triangle C = \boxed{\{\varnothing, \{1,2\}, 1\}}$

(f) True or False: $B \in A$.

$\boxed{\text{True}}$ $B = \{2,1\} = \{1,2\} \in A$

(g) True or False: $B \subseteq A$.

$\boxed{\text{False}}$ $1 \in B$ but $1 \notin A$

(h) True or False: $B \in \mathcal{P}(C)$.

$\boxed{\text{False}}$ $B \in \mathcal{P}(C) \Leftrightarrow B \subseteq C$

But $2 \in B$ and $2 \notin C$ so $B \not\subseteq C$.

(i) True or False: $A \subseteq \mathcal{P}(B)$.

$\boxed{\text{True}}$ In fact, $A = \mathcal{P}(B)$.

(j) True or False: $B \cap C = 1$.

$\boxed{\text{False}}$ $B \cap C$ is a set, $\underline{1}$ is an integer

2. **[4 parts, 2 points each]** Let $A = \{(1,1)\}$ and $B = \{(1,2),(3,3)\}$.

(a) Determine $|A|$ and $|B|$.

$|A| = 1$ (One ordered pair)

$|B| = 2$ (Two ordered pairs)

(b) Determine $A \times B$.

$A \times B = \boxed{\{ ((1,1),(1,2)),\ ((1,1),(3,3)) \}}$

(c) Determine $\mathcal{P}(A)$.

Let $x = (1,1)$; $A = \{x\}$.

$\mathcal{P}(A) = \{\emptyset, \{x\}\} = \boxed{\{\emptyset, \{(1,1)\}\}}$

(d) Determine B^0.

$B^0 = \boxed{\{()\}}$

3. **[3 parts, 3 points each]** Let $C = \{1, 2, \ldots, 10\}$. In terms of C, give the set that best models the space of all possibilities for each situation described below. For example, if the situation is "Joe thinks of a number between 1 and 10", then the appropriate set is just C.

(a) Joe and Alice both think of numbers between 1 and 10.

Possibilities: a pair of numbers in C; first number for Joe, second for Alice.

$\boxed{C \times C}$ or $\boxed{C^2}$

(b) A computer initializes a game of minesweeper on a (10×10) board, where each of the 100 cells might or might not contain a bomb.

The cells correspond to an ordered pair in $C \times C$.

The bombs are at some subset of $C \times C$.

Space of all subsets: $\boxed{\mathcal{P}(C \times C)}$ or $\boxed{\mathcal{P}(C^2)}$

(c) A pizza restaurant offers 10 different toppings, numbered from 1 through 10 on the menu. Two people order pizzas from the restaurant, each with a particular set of toppings.

Each pizza order corresponds to a subset of C, so to an element in $\mathcal{P}(C)$.

Two pizza orders: $\boxed{\mathcal{P}(C) \times \mathcal{P}(C)}$ or $\boxed{(\mathcal{P}(C))^2}$.

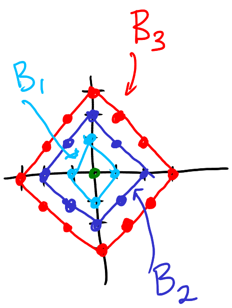
4. Are the following sets countable or not? If countable, then describe how to enumerate the set. If uncountable, then justify your answer by adapting Cantor's Diagonalization Argument.

   (a) [**5 points**] The set of integers $\mathbb{Z}$.

$\boxed{\text{Countable:}}$ $0, -1, 1, -2, 2, -3, 3, \ldots$

We enumerate the elements of $\mathbb{Z}$ in order of absolute value.

   (b) [**5 points**] The set $\mathbb{Z} \times \mathbb{Z}$.



$\boxed{\text{Countable:}}$ partition $\mathbb{Z}\times\mathbb{Z}$ into blocks $B_0, B_1, \ldots$ so that

$B_n = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} : |x| + |y| = n\}$ and list the elements of the blocks in order. Since each block is finite (in fact $|B_n| = 4n$ for $n \geq 1$), the elements of each block eventually appear.

Our enumeration begins:

$$\underbrace{(0,0)}_{B_0}, \underbrace{(1,0), (0,1), (-1,0), (0,-1)}_{B_1}, \underbrace{(2,0), (1,1), (0,2), (-1,1), (-2,0), (-1,-1), (0,-2), (1,-1)}_{B_2}, \ldots$$

   (c) [**6 points**] The set $\{A\colon\ A \text{ is a finite set of integers}\}$.

$\boxed{\text{Countable}}$. Let $B_n = \{A : A \text{ is a subset of } \{-n, -(n-1), \ldots, 0, \ldots, n-1, n\}\}$

$= \mathcal{P}(\{-n, -(n-1), \ldots, 0, \ldots, n-1, n\})$.

Since $|B_n| = 2^{2n+1}$ and every finite set of integers is in some block B_n, we enumerate the finite sets of integers by block, crossing out duplicates. Our enumeration begins:

$$\underbrace{\emptyset, \{0\}}_{B_0 = \mathcal{P}(\{0\})}, \underbrace{\cancel{\emptyset}, \{-1\}, \cancel{\{0\}}, \{1\}, \{-1,0\}, \{-1,1\}, \{0,1\}, \{-1,0,1\}}_{B_1 = \mathcal{P}(\{-1,0,1\})}, \underbrace{\cancel{\emptyset}, \{-2\}, \cancel{\{-1\}}, \cancel{\{0\}}, \cancel{\{1\}}, \{2\}, \ldots}_{B_2 = \mathcal{P}(\{-2,-1,0,1,2\})}$$

5. A 4-digit ATM pin is selected at random. Let A be the event that the first digit is even, and let B be the event that at least one of the digits is 7.

   (a) [**2 points**] What is the sample space Ω? What is $|\Omega|$?

$$\Omega = \{0,1,2,\dots,9\}^4$$

$$|\Omega| = 10^4$$

   (b) [**6 points**] Determine $\Pr(A)$ and $\Pr(B)$.

$$|A| = \underbrace{5}_{\text{first digit even}} \cdot \overbrace{10\cdot 10\cdot 10}^{\text{any digit}} = 5\cdot 10^3 \;; \quad \Pr(A) = \frac{|A|}{|\Omega|} = \frac{5\cdot \cancel{10^3}}{\cancel{10^4}} = \frac{5}{10} = \boxed{\frac{1}{2}}$$

$$|B| = 10^4 - (\#\text{ pins with no } 7) = 10^4 - \underbrace{9\cdot 9\cdot 9\cdot 9}_{\text{any digit except } 7} = 10^4 - 9^4 = 3439$$

$$\Pr(B) = \frac{|B|}{|\Omega|} = \frac{10^4 - 9^4}{10^4} = \boxed{\frac{3439}{10000}} = \boxed{0.3439}$$

   (c) [**6 points**] Determine $\Pr(A|B)$ and $\Pr(B|A)$.

Need to count $A\cap B$: pins with first digit even <u>and</u> at least one 7.

$$|A\cap B| = \underbrace{5}_{\text{even first digit}} \cdot (\#3 \text{ digit pins w/ } \geq 1 \text{ seven}) = 5\cdot(\underbrace{10^3}_{\text{all 3 digit pins}} - \underbrace{9^3}_{\text{3 digit pins with no 7's}}) = 1355$$

$$\Pr(A|B) = \frac{\Pr(A\cap B)}{\Pr(B)} = \frac{|A\cap B|/|\Omega|}{|B|/|\Omega|} = \frac{|A\cap B|}{|B|} = \boxed{\frac{1355}{3439}} \approx 0.3940$$

$$\Pr(B|A) = \frac{\Pr(B\cap A)}{\Pr(A)} = \frac{|A\cap B|}{|A|} = \frac{1355}{5\cdot 10^3} = \frac{1355}{5000} = \boxed{\frac{271}{1000}} = \boxed{0.271}$$

   (d) [**2 points**] Are the events A and B pos. correlated, neg. correlated, or independent?

Since $\Pr(A|B) \approx 0.3940 < \frac{1}{2} = \Pr(A)$, we have that

A and B are $\boxed{\text{negatively correlated}}$

6. Recall that a standard deck of cards has one card for each rank/suit pair, where the ranks are [ace, 2 through 10, jack, queen, king], and the suits are [clubs, hearts, diamonds, spades]. A 5-card poker hand is dealt from a freshly shuffled deck. Let A be the event that the hand contains exactly one ace and let B be the event that the hand contains exactly one spade.

   (a) [**5 points**] Determine $\Pr(A)$ and $\Pr(B)$.

$$\Pr(A) = \frac{|A|}{|\Omega|} = \frac{\overbrace{\binom{4}{1}}^{\text{choose ace}} \cdot \overbrace{\binom{48}{4}}^{\text{choose non-aces}}}{\binom{52}{5}} = \boxed{\frac{4\binom{48}{4}}{\binom{52}{5}}} \approx 0.2995$$

$$\Pr(B) = \frac{|B|}{|\Omega|} = \frac{\overbrace{\binom{13}{1}}^{\text{choose spade}} \cdot \overbrace{\binom{39}{4}}^{\text{choose non spades}}}{\binom{52}{5}} = \boxed{\frac{13\binom{39}{4}}{\binom{52}{5}}} \approx 0.4114$$

   (b) [**5 points**] Determine $\Pr(A|B)$.

Need to count $A \cap B$, the hands with exactly one ace and exactly one spade.

Two groups: hands in $A \cap B$ with ace of spades: $\underbrace{1}_{\text{ace of spades}} \cdot \underbrace{\binom{52-16}{4}}_{\text{remaining cards}} = \binom{36}{4}$

hands in $A \cap B$ without ace of spades: $\underbrace{3}_{\text{ace}} \cdot \underbrace{12}_{\text{spade}} \cdot \underbrace{\binom{52-16}{3}}_{\text{remaining cards}} = 36 \cdot \binom{36}{3}$

$$\text{So } \Pr(A|B) = \frac{|A \cap B|}{|B|} = \boxed{\frac{\binom{36}{4} + 36\binom{36}{3}}{13 \cdot \binom{39}{4}}} \approx 0.2955$$

7. [**6 points**] Suppose that 90% of the population exceeds the speed limit regularly when driving. A sociology study finds that, when asked, a non-speeder will always deny speeding. However, 20% of the time, a speeder will falsely claim not to speed. John claims that he is not a speeder. What is the probability that John is not a speeder?

A: John is not a speeder, B: John claims not to speed

$$\Pr(A|B) = \frac{\Pr(A)}{\Pr(B)} \cdot \Pr(B|A) = \frac{\frac{1}{10}}{\Pr(B)} \cdot 1 = \frac{1}{10\left(\Pr(B|A)\cdot\Pr(A) + \Pr(B|\bar{A})\cdot\Pr(\bar{A})\right)}$$

$$= \frac{1}{10\left(1 \cdot \frac{1}{10} + \frac{2}{10}\cdot\frac{9}{10}\right)} = \frac{1}{1 + \frac{18}{10}} = \frac{1}{\frac{28}{10}} = \frac{10}{28} = \boxed{\frac{5}{14}} \approx 0.3571$$

8. **[3 parts, 3 points each]** Let $\Sigma = \{0,1\}$. We define the following languages:

$$A = \{w \in \Sigma^* : \text{ } w \text{ has even length}\}$$
$$B = \{w \in \Sigma^* : \text{ } w \text{ has an odd number of ones}\}$$

(a) Give an example of a string in $A \cap B$.

Need even length and an odd number of ones. So possible answers:

01, 10, 1000, 0100, 1101, etc

(b) Describe the relationship between A and AA. Justify your answer.

We have $\boxed{AA = A.}$ Every string in AA is made by concatenating two even length strings, so $AA \subseteq A$. Conversely $A \subseteq AA$ because if $x \in A$ then $\lambda x \in AA$ and $\lambda x = x$.

(c) Is there a string in $B \cap BB$? Give an example or explain why not.

$\boxed{\text{No.}}$ Every string in BB is the concatenation of two strings with an odd number of 1's, and so every string in BB has an even number of 1's. Every string in B has an odd number of 1's, so $BB \cap B = \emptyset$.

9. Let $\Sigma = \{0,1\}$. We define a language A over Σ recursively as follows.

(1) $\lambda \in A$

(2) If $x \in A$, then $0x \in A$.

(3) If $x \in A$, then $1x1 \in A$.

(a) **[3 points]** Is $0110 \in A$? Explain why or why not.

$\boxed{\text{No}}$ The only way for $0110 \in A$ would by by rule 2, which would require $110 \in A$. But none of the rules makes $110 \in A$ possible.

(b) **[3 points]** Is $1001 \in A$? Explain why or why not.

$\boxed{\text{Yes}}$. By (1), $\lambda \in A$. By (2), $0 = 0\lambda \in A$. By (2) again, $00 \in A$. By (3), since $00 \in A$, we have $1001 \in A$.

(c) **[5 bonus points (tricky, finish others first)]** Give a simple description of A.

$\boxed{A = \{ w1^n : n \geq 0 \text{ and } w \text{ has } n \text{ ones} \}}$. Equivalently, A is the set of strings with an even number of ones, at least half of which form a suffix.

Scratch Paper