Name: Solutions

Directions: Show all work. Answers without work generally do not earn points. Unless stated otherwise, answers may be left in terms of factorials and binomial coefficients. Your answers should not involve sums with more than a few terms.

1. **[3 parts, 5 points each]** A bakery offers cake and pie slices. A cake slice can be ordered in 3 different sizes (small, medium, large), 4 different batters (vanilla, chocolate, red velvet, strawberry), and 3 different frosting types (butter, whipped cream, cream cheese). There are 7 different types of pie slices (apple, blueberry, strawberry, cherry, key lime, pecan, pumpkin).

   (a) How many ways are there to order a cake slice from the bakery?

   Rule of product

   $$\underset{\text{size}}{3} \cdot \underset{\text{batter}}{4} \cdot \underset{\text{frosting}}{3} = \boxed{36}$$

   (b) Mike and Mary visit the shop. How many ways can the pair order desserts, assuming Mike wants to order a cake slice and Mary wants to order a pie slice?

   Rule of Product.

   1. Mike orders cake $n_1 = 36$ (From part (a))
   2. Mary orders pie $n_2 = 7$

   $36 \cdot 7 = \boxed{252}$

   (c) Now suppose that Mike still wants a cake slice and Mary still wants a pie slice, but they want to avoid a situation where Mike gets cake with strawberry batter and, at the same time, Mary gets the strawberry pie. How many ways are there for the pair to order dessert?

   Count the complement: Mike gets strawberry batter: $\underset{\text{size}}{3} \cdot \underset{\text{batter}}{1} \cdot \underset{\text{frosting}}{3} = 9.$

   Mary gets strawberry pie: $1.$

   Bad Orders: $9 \cdot 1 = 9$

   Good orders: $252 - 9 = \boxed{243}$

2. **[3 parts, 5 points each]** How many 4-digit ATM pin numbers are there:

   (a) that begin with an even number? (For example 0123 and 0000 count, but 1246 does not.)

   Rule of Product:

   $$\underset{\text{1st digit even}}{5} \cdot \underbrace{10 \cdot 10 \cdot 10}_{\text{2nd to 4th arbitrary}} = 5 \cdot 10^3 = \boxed{5000}$$

   (b) that have a 4 in the second or third positions? (For example, 1401, 1041, and 1441 all count, but 4114 does not.)

   Count complement. No 4 in pos. 2 or 3: $10 \cdot 9 \cdot 9 \cdot 10 = 8100$

   So #(pins w/ 4 in 2nd or 3rd positions) $= 10^4 - 8100 = \boxed{1900}$

   (c) that contain at least 3 distinct digits? (For example, 1234 and 7989 count, but 7979 and 7999 do not.)

   Rule of Sum. All digits distinct: $10 \cdot 9 \cdot 8 \cdot 7$

   One pair of repeated digits: $\underset{\text{Locations for repetition}}{\binom{4}{2}} \cdot \underset{\text{repeated digit value}}{10} \cdot \underbrace{9 \cdot 8}_{\text{value of digits appearing once}}$

   $$(10 \cdot 9 \cdot 8 \cdot 7) + 10 \cdot 9 \cdot 8 \cdot \binom{4}{2} = 10 \cdot 9 \cdot 8(7+6) = \boxed{9360}$$

3. **[5 points]** A bank's safe has a lock with a dial with 60 numbered positions, from 0 through 59 inclusive. The combination to the safe is a list of 5 numbers, such as 32-0-25-32-41. Although repetitions in the combination are allowed, consecutive repetitions are not. How many combinations are possible?

   Rule of product

   $$\boxed{\underset{\text{first number}}{60} \cdot \underset{\text{numbers 2 through 5}}{(59)^4}} = \boxed{727,041,660}$$

4. **[3 parts, 5 points each]** Poker hands. Recall that a standard deck has 52 cards: one for each suit/rank pair, where the 4 suits are spades, hearts, diamonds, clubs and the 13 ranks are ace, 2 through 10, jack, queen, and king. A poker hand is a set of 5 cards from the deck.

   (a) How many poker hands contain the five of hearts, $5H$?

Start with 5H.
Choose 4 cards from the remaining 51:

$$\boxed{\binom{51}{4}} = \boxed{249,900}$$

   (b) How many poker hands have at least one king?

Count complement.

$52-4 = 48$ non-kings

$\binom{48}{5}$ hands with no kings

$$\#\binom{\text{hands with}}{\geq 1 \text{ king}} = \boxed{\binom{52}{5} - \binom{48}{5}} = \boxed{886,656}$$

   (c) How many poker hands have cards from exactly two suits? For example, $\{4C, 7C, 8C, 10C, AD\}$ counts but $\{4C, 7C, 8C, 10C, AC\}$ does not.

clubs + #spades

(Hands where all cards are clubs or spades) $= \binom{26}{5}$

(Hands where all cards are clubs <u>or</u> all cards are spades) $= 2\binom{13}{5}$

(Hands where the suits that appear are exactly clubs and spades) $= \binom{26}{5} - 2\binom{13}{5}$

(Hands from exactly two suits) $= \boxed{\binom{4}{2} \cdot \left[\binom{26}{5} - 2\binom{13}{5}\right]} = \boxed{379,236}$

choose two suits

5. **[4 parts, 5 points each]** Find the number of integer solutions to the following equations with the given constraints.

(a) $x_1 + x_2 + x_3 + x_4 + x_5 = 82$, each $x_i \geq 0$

$\underbrace{* \cdots *}_{82 \text{ stars}} \; \underbrace{||||}_{4 \text{ bars}}$ $\qquad \boxed{\binom{86}{4}} = \boxed{2{,}123{,}555}$

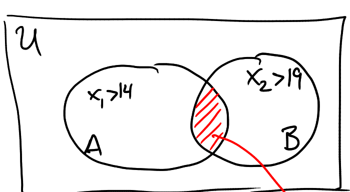
(b) $x_1 + x_2 + x_3 + x_4 + x_5 = 82$, each $x_i \geq 6$

Reserve $6 \cdot 5$ or 30 coins.

$\hat{x}_1 + \cdots + \hat{x}_5 = 52$

$\underbrace{* \cdots *}_{52 \text{ stars}} \; \underbrace{||||}_{4 \text{ bars}} \Rightarrow \boxed{\binom{56}{4}} = \boxed{367{,}290}$

(c) $x_1 + x_2 + x_3 + x_4 + x_5 = 82$, each $x_i \geq 0$, $x_1 \leq 14$, and $x_2 \leq 19$. (Hint: draw a Venn Diagram.)



$\mathcal{U}$: all solns to $x_1 + \cdots + x_5 = 82$, $x_i \geq 0$: $\binom{86}{4}$ (part a)

A: solns in $\mathcal{U}$ with $x_1 \geq 15$: $\hat{x}_1 + \cdots + \hat{x}_5 = 67$ 67 stars; 4 bars $\Rightarrow \binom{71}{4}$

B: solns in $\mathcal{U}$ with $x_2 \geq 20$: $\hat{x}_1 + \cdots + \hat{x}_5 = 62$ 62 stars; 4 bars $\Rightarrow \binom{66}{4}$

$A \cap B$: solns in $\mathcal{U}$ with $x_1 \geq 15$, $x_2 \geq 20$: $\hat{x}_1 + \cdots + \hat{x}_5 = 47$ 47 stars; 4 bars $\Rightarrow \binom{51}{4}$.

So # good solns:

$\boxed{\binom{86}{4} - \binom{71}{4} - \binom{66}{4} + \binom{51}{4}} = \boxed{681{,}100}$

(d) $x_1 + \cdots + x_n = 4n$ with each $x_i \geq 2$.

Reserve $2n$ stars:

$\hat{x}_1 + \cdots + \hat{x}_n = 4n - 2n = 2n$

$\underbrace{* \cdots *}_{2n \text{ stars}} \; \underbrace{|| \cdots |}_{n-1 \text{ bars}} \Rightarrow \binom{2n + (n-1)}{n-1} = \boxed{\binom{3n-1}{n-1}}$

6. **[3 parts, 5 points each]** Find the coefficient in the following. **You may use factorials in your answer, but not binomial or multinomial coefficients.**

(a) x^6y^3 in $(2x+3y)^9$

$A = 2x, \quad B = 3y$

$(A+B)^9 \Rightarrow \binom{9}{3} A^6 B^3 = \binom{9}{3}(2x)^6(3y)^3$

$= \binom{9}{3} \cdot 2^6 \cdot 3^3 \cdot x^6 y^3$

So coefficient is $\boxed{\binom{9}{3} \cdot 2^6 \cdot 3^3} = \boxed{145,152}$

(b) $w^3x^2y^6z^4$ in $(w+x+y+z)^{15}$

Multinomial Thm: $(w+x+y+z)^{15} \Rightarrow \binom{15}{3,2,6,4} w^3 x^2 y^6 z^4$

So coefficient is $\binom{15}{3,2,6,4} = \boxed{\frac{(15)!}{(3!)(2!)(6!)(4!)}} = \boxed{6,306,300}$

(c) x^8 in $(x-3y+2)^{11}$.

$A = x, \quad B = -3y, \quad C = 2$

$(A+B+C)^{11} \Rightarrow \binom{11}{8,0,3} \cdot A^8 B^0 C^3 = \binom{11}{8,0,3} x^8 \cdot \cancel{(-3y)^0}^{\,1} \cdot (2)^3$

$= \binom{11}{8,0,3} \cdot 2^3 \cdot x^8$

So coefficient is $\binom{11}{8,0,3} \cdot 2^3 = 8 \cdot \frac{(11)!}{8! \cdot 3!} = \boxed{1320}$

7. **[5 points]** A donut shop makes a total of 15 kinds of donuts. The most popular item for sale is a variety box, which is filled with 12 donuts of the customer's choice. The box may have all distinct donut types, or a single type of donut, or other combinations. How many different variety boxes are possible? (Note: the location of donuts within the box is not considered important, so shuffling the donuts within a box does not give a new type of box.)

For $1 \le i \le 15$, let x_i be # of donuts of type i in the box.

$\#$ boxes $=$ # solns to $x_1 + \cdots + x_{15} = 12$, each $x_i \geq 0$

$$\underbrace{* \cdots *}_{12 \text{ stars}} \quad \underbrace{|| \cdots |}_{14 \text{ bars}} \Rightarrow \binom{26}{14} \text{ or } \binom{26}{12}$$

So # boxes $= \boxed{\binom{26}{12}} = \boxed{9,657,700}$

8. **[5 points]** How many lists of positive integers sum to 10? For example, $(4, 2, 3, 1)$, $(1, 2, 3, 4)$, $(2, 2, 2, 2, 2)$, and (10) all count, but $(5, 1, 0, 4)$ does not (since all entries in the list must be positive).

From HW: $\underbrace{\overbrace{o \; o \cdots o}^{9 \text{ gaps}}}_{10 \text{ circles}}$ between each gap, choose to insert a dividing bar or choose not to

$$(4, 2, 3, 1) \Leftrightarrow o \; o \; o \; o \,|\, o \; o \,|\, o \; o \; o \,|\, o$$

So # lists $= 2^{10-1} = \boxed{2^9} = \boxed{512}$.

9. **[5 points]** Evaluate the sum $\sum_{k=0}^{n} \binom{n}{k} 4^k$.

Binomial Thm: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

with $x=1$, $y=4$: $(1+4)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 4^k = \sum_{k=0}^{n} \binom{n}{k} 4^k$

So sum evaluates to $\boxed{5^n}$.