Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[4 parts, 1 point each]** Let $A = \{(2,4), \{4,2\}, 3\}$, $B = \{3, (4,2)\}$, and $C = \{1,2,3\} \times \{3,4\}$.

   (a) Determine the sizes of A, B, and C.

   $|A| = 3$ (one ordered pair, one set, one integer)

   $|B| = 2$ (one integer, one ordered pair)

   $|C| = 3 \cdot 2 = 6$ (6 ordered pairs)

   (b) Find $A \cap B$.

   $A \cap B = \boxed{\{3\}}$. (Note that $(4,2) \neq (2,4)$ and $(4,2) \neq \{4,2\}$

   (c) Find B^2.

   $B^2 = \{(b_1, b_2) : b_1 \in B \text{ and } b_2 \in B\}$

   $= \{(3,3), (3,(4,2)), ((4,2),3), ((4,2),(4,2))\}$

   (d) Find C^0.

   $C^0 = \{\text{lists of length } 0, \text{ all members in } C\}$

   $= \boxed{\{()\}}$

2. **[2 points]** Is it true or false that for all sets A, B, C, we have that $(A \times B) \times C = A \times (B \times C)$? If true, then explain why this is true, and if false, then give an example of sets A, B, C where $(A \times B) \times C \neq A \times (B \times C)$.

   Although these sets are closely related, they are <u>not</u> generally equal, so this is $\boxed{\text{false}}$. For example: let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$.

   Now $(A \times B) \times C = \{(1,2)\} \times \{3\} = \{((1,2),3)\}$

   $A \times (B \times C) = \{1\} \times \{(2,3)\} = \{(1,(2,3))\}$.

3. [**2 points**] Is the set $\mathbb{N}^5$ countable or not? Justify your answer. Note: Multiple Solns Possible.

Yes, $\mathbb{N}^5$ is $\boxed{\text{countable.}}$ We can enumerate $\mathbb{N}^5$ in blocks where for $n \geq 0$, the n^{th} block contains all 5-tuples $(x_1, \ldots, x_5)$ with entries summing to n:

$$\underbrace{(0,0,0,0,0)}_{\text{block } 0}, \underbrace{(1,0,0,0,0), \ldots, (0,0,0,0,1)}_{\text{block } 1}, \underbrace{(2,0,0,0,0), (1,1,0,0,0), \ldots}_{\text{block } 2}, \ldots$$

By stars and bars, block n contains $\binom{n+4}{4}$ entries. Since each block is finite, each $(x_1, \ldots, x_5) \in \mathbb{N}^5$ eventually appears.

4. [**2 points**] Let A be the set whose members are the subsets of the positive integers. For example, the following sets are members of A: $\{1, 3, 5, 7, \ldots\}$, $\{n : n \text{ is prime}\}$, $\{1, 2, 3, 4, 5\}$, $\varnothing$, and $\{1, 4, 9, 25, 36, \ldots\}$. Let $S_1, S_2, S_3, \ldots$ be a list of members of A. Adapt Cantor's diagonalization argument to construct a set D which does not appear on the list.

We set D so that D and S_1 disagree on membership of 1,
D and S_2 disagree on membership of 2,
etc.

That is, we set $\boxed{D = \{n : n \notin S_n\}}$.